8.13

We need N for the Low-temp Fermi Gas' chemical potential, and U to find C_V (and hence S). Let's find N and U first.

→ By 8.1.2, $N = \sum_{\varepsilon} \frac{1}{z^{-1}e^{\beta\varepsilon}+1} = \int_{\varepsilon=0}^{\infty} \frac{a(\varepsilon)\,d\varepsilon}{z^{-1}e^{\beta\varepsilon}+1} = \int_0^{\infty} \frac{a(\varepsilon)\,d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$

Make 2 u-substitutions:

$x = \beta\varepsilon \rightarrow \varepsilon = x/\beta \Rightarrow d\varepsilon = dx/\beta$ and $\xi = \beta\mu$

$$N = \frac{1}{\beta}\int_0^{\infty} \frac{a(x/\beta)\,dx}{e^{x-\xi}+1} = \frac{1}{\beta}\left(\int_0^{\xi} a(x/\beta)\,dx + \frac{\pi^2}{6}\left(\frac{da(x/\beta)}{dx}\right)_{x=\xi} + \cdots\right)$$

By E-18

Transform back to μ and ε:

$$N = \int_0^{\mu} a(\varepsilon)\,d\varepsilon + \frac{1}{\beta^2}\frac{\pi^2}{6}\left(\frac{da(\varepsilon)}{d\varepsilon}\right)_{\varepsilon\beta=\mu\beta} + \cdots$$

For low temperatures $\mu \approx \varepsilon_F$ for 8.1.34

$$\Rightarrow N = \int_0^{\varepsilon_F} a(\varepsilon)\,d\varepsilon + \frac{1}{\beta^2}\frac{\pi^2}{6}\left(\frac{da(\varepsilon)}{d\varepsilon}\right)_{\varepsilon=\varepsilon_F}$$

Also, note that $\mu - \varepsilon_F \approx 0$, so we can freely add $(\mu-\varepsilon_F)a(\varepsilon_F)$

$$(1)\quad N = \int_0^{\varepsilon_F} a(\varepsilon)\,d\varepsilon + \frac{1}{\beta^2}\frac{\pi^2}{6}\left(\frac{da(\varepsilon)}{d\varepsilon}\right)_{\varepsilon=\varepsilon_F} + (\mu-\varepsilon_F)\,a(\varepsilon_F)$$

→ $U = -\frac{\partial}{\partial\beta}\ln Q = -\frac{\partial}{\partial\beta}\int_0^{\infty}\ln(1+ze^{-\beta\varepsilon})\,a(\varepsilon)\,d\varepsilon = -\int_0^{\infty}\frac{z\,\varepsilon\,e^{-\beta\varepsilon}a(\varepsilon)}{1+e^{-\beta\varepsilon}z}\,d\varepsilon$

$$= \int_0^{\infty}\frac{\varepsilon\,a(\varepsilon)}{z^{-1}e^{\beta\varepsilon}+1}\,d\varepsilon$$

Now use the same substitutions as above, and use E-18 exactly as before:

$$U = \frac{1}{\beta^2}\left(\int_0^{\xi} x\,a(x/\beta)\,dx + \frac{\pi^2}{6}\left(\frac{d}{dx}x\,a(x/\beta)\right)_{x=\xi} + \cdots\right)$$

$$= \int_0^{\mu}\varepsilon\,a(\varepsilon)\,d\varepsilon + \frac{1}{\beta^3}\frac{\pi^2}{6}\left(\frac{d}{d\varepsilon}\varepsilon\beta\,a(\varepsilon)\right)_{\varepsilon=\mu}$$

So, letting $\mu \approx \varepsilon_F$, and taking the product-rule derivative:

$$(2)\quad U = \int_0^{\varepsilon_F}\varepsilon\,a(\varepsilon)\,d\varepsilon + \frac{1}{\beta^2}\frac{\pi^2}{6}\varepsilon_F\left(\frac{da(\varepsilon)}{d\varepsilon}\right)_{\varepsilon=\varepsilon_F} + \frac{1}{\beta^2}\frac{\pi^2}{6}a(\varepsilon_F) + (\mu-\varepsilon_F)\,\varepsilon_F\,a(\varepsilon_F)$$

Ok. Now let's find μ.

By 8.1.20, $\int_0^{\varepsilon_F} a(\varepsilon)\,d\varepsilon = N$. Make this substitution in ①:

$$N \approx N + \frac{1}{\beta^2}\frac{\pi^2}{6}\left(\frac{da(\varepsilon)}{d\varepsilon}\right)_{\varepsilon=\varepsilon_F} + (\mu-\varepsilon_F)a(\varepsilon_F)$$

$$(\varepsilon_F-\mu)a(\varepsilon_F) \approx \frac{1}{\beta^2}\frac{\pi^2}{6}\left(\frac{\partial a(\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=\varepsilon_F} \Rightarrow \mu \approx \varepsilon_F - \frac{1}{\beta^2 a(\varepsilon_F)}\frac{\pi^2}{6}\left(\frac{da(\varepsilon)}{d\varepsilon}\right)_{\varepsilon=\varepsilon_F}$$

$$\Rightarrow \mu \approx \varepsilon_F\left[1 - \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\left(\frac{\frac{1}{a(\varepsilon)}da(\varepsilon)}{\frac{1}{\varepsilon}d\varepsilon}\right)_{\varepsilon=\varepsilon_F}\right.$$

Now, note that $\frac{1}{x}dx = d\ln(x)$.

$$\therefore \boxed{\mu \approx \varepsilon_F\left[1 - \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\left(\frac{d\ln(a(\varepsilon))}{d\ln(\varepsilon)}\right)_{\varepsilon=\varepsilon_F}\right.}$$

we then use U to find C_V. (use $C_V = \left(\frac{dU}{dT}\right)_V$)

Plug the expression we just found into ②:

$$U = \int_0^{\varepsilon_F}\varepsilon a(\varepsilon)d\varepsilon + \frac{k^2T^2\pi^2}{6}\left(\varepsilon_F\frac{da(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=\varepsilon_F} + a(\varepsilon_F)\right) - \varepsilon_F^2 a(\varepsilon_F) + \varepsilon_F^2 a(\varepsilon_F)\left[1-\frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\frac{\partial\ln a}{\partial\ln\varepsilon}\Big|_{\varepsilon=\varepsilon_F}\right]$$

Since we are taking a derivative w/ respect to T, lets ignore the terms without T:

$$U = \text{"constants"} + \frac{k^2T^2\pi^2}{6}\left[\varepsilon_F\frac{da(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=\varepsilon_F} + a(\varepsilon_F) - a(\varepsilon_F)\left(\frac{\partial\ln a(\varepsilon)}{\partial\ln\varepsilon}\right)\Big|_{\varepsilon=\varepsilon_F}\right]$$

Undo the trick that we used before: $a(\varepsilon)d\ln(a(\varepsilon)) = da(\varepsilon)$
$\varepsilon\, d\ln(\varepsilon) = d\varepsilon$

$$U = \text{"constats"} + \frac{k^2T^2\pi^2}{6}\left[\cancel{\varepsilon_F\frac{a(\varepsilon)d\ln(a(\varepsilon))}{\varepsilon_F d\ln(\varepsilon)}}\Big|_{\varepsilon=\varepsilon_F} + a(\varepsilon_F) - \cancel{a(\varepsilon_F)\frac{d\ln(a(\varepsilon))}{d\ln\varepsilon}}\Big|_{\varepsilon=\varepsilon_F}\right]$$

$$= \text{"constats"} + \frac{kT^2\pi^2}{6}a(\varepsilon_F)$$

$$\therefore \boxed{C_V \approx \frac{dU}{dT} \approx \frac{k^2T\pi^2}{3}a(\varepsilon_F)}$$

Entropy

By 1.3.17, $C_V = T\left(\frac{\partial S}{\partial T}\right)_{N,V} \Rightarrow \frac{k^2\pi^2}{3}a(\varepsilon_F) = \frac{dS}{dT}$ integrate to get: $\boxed{S \approx \frac{k^2T\pi^2}{3}a(\varepsilon_F)}$

Now we consider the case where $\varepsilon \propto p^s$ (i.e. $\varepsilon = Ap^s$) in n-dimensions

Let's find the density of states.

$$\Sigma(p) = \frac{1}{h^n}\int d^nq\,d^np = \frac{V_n}{h^n}V_n(p) = \frac{V_n}{h^n}\frac{\pi^{n/2}}{(n/2)!}p^n \text{ by appendix C.7a.}$$

$$g(p)dp = \frac{d\Sigma(p)}{dp}dp = \frac{nVp^{n-1}}{h^n}\frac{\pi^{n/2}}{(n/2)!}dp$$

Now $\varepsilon = Ap^s \Rightarrow p = \left(\frac{\varepsilon}{A}\right)^{1/s}$, so $dp = d\varepsilon\frac{dp}{d\varepsilon} = d\varepsilon\frac{1}{sA^{1/s}}\varepsilon^{1/s-1}$

and $p^{n-1} = \left(\frac{\varepsilon}{A}\right)^{\frac{n}{s}-\frac{1}{s}}$ $\Rightarrow a(\varepsilon)d\varepsilon \sim (\varepsilon)^{\frac{n}{s}-\frac{1}{s}}\cdot\varepsilon^{1/s-1}d\varepsilon$

Thus $a(\varepsilon)d\varepsilon \sim \varepsilon^{\frac{n}{s}-1} d\varepsilon$

Thus for n particles, and $p \propto \varepsilon^s$,

$$C_v \approx S \approx \frac{k^2 T \pi^2}{3} a(\varepsilon_F) \sim \frac{k^2 T \pi^2}{3} \varepsilon^{\frac{n}{s}-1}$$

For μ, we need $\left.\frac{\partial \ln(a(\varepsilon))}{\partial \ln \varepsilon}\right|_{\varepsilon=\varepsilon_F} = \frac{\varepsilon_F}{a(\varepsilon_F)} \left.\frac{\partial a(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon_F}$.

$$= \frac{\cancel{\varepsilon_F}}{\cancel{\varepsilon_F^{\frac{n}{s}-1}}} \left(\frac{n}{s}-1\right) \cancel{\varepsilon_F^{\frac{n}{s}-2}} = \frac{n}{s}-1$$

$$\Rightarrow \mu \approx \varepsilon_F \left[1 - \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2 \left(\frac{n}{s}-1\right)\right]$$

We are expected to examine the following cases:

$\underline{s=1, n=2}$: $\quad C_v \approx S \approx \frac{k^2 T \pi^2 \varepsilon}{3}$, $\quad \mu \approx \varepsilon_F \left[1 - \frac{\pi^2}{6}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$

$\underline{s=1, n=3}$: $\quad C_v \approx S \approx \frac{k^2 T \pi^2 \varepsilon^2}{3}$, $\quad \mu \approx \varepsilon_F \left[1 - \frac{\pi^2}{3}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$

$\underline{s=2, n=2}$: $\quad C_v \approx S \approx \frac{k^2 T \pi^2}{3}$, $\quad \mu \approx \varepsilon_F$

$\underline{s=2, n=3}$: $\quad C_v \approx S \approx \frac{k^2 T \pi^2 \varepsilon^{1/2}}{3}$, $\quad \mu \approx \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$

$\frac{10}{10}$