8.12

Lemma 1: $(1+z_F)(1-z_B)=1$

Proof: First we need the 2-D free particle density of states:

$\varepsilon = \frac{p^2}{2m} \Rightarrow p^2 = 2m\varepsilon \qquad \frac{d\varepsilon}{dp} = \frac{p}{m} \Rightarrow dp = \frac{m\,d\varepsilon}{p}$

$\frac{1}{h^2}\int d^2q\,d^2p = \frac{L^2}{h^2}\int 2\pi p\,dp = \frac{L^2}{h^2}\int 2\pi m\,d\varepsilon = \frac{2\pi L^2 m}{h^2}\int d\varepsilon \Rightarrow a(\varepsilon)\,d\varepsilon = \frac{2\pi m L^2}{h^2}\,d\varepsilon$

Bose: Then, since $N_B = \sum_\varepsilon \frac{1}{z^{-1}e^{\beta\varepsilon}-1}$ by 7.1.2, $\frac{N_B}{L^2} = \int \frac{a(\varepsilon)}{z^{-1}e^{\beta\varepsilon}-1}\,d\varepsilon$

$\Rightarrow \frac{N_B}{L^2} = \frac{2\pi m}{h^2}\cdot\int_0^\infty \frac{d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1} = \frac{-2\pi m}{\beta h^2}\ln(1-z_B)$

Fermi: Next $N_F = \sum_\varepsilon \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \Rightarrow \frac{N_F}{L^2} = \int \frac{a(\varepsilon)}{z^{-1}e^{\beta\varepsilon}+1}\,d\varepsilon = \frac{2\pi m}{h^2}\int_0^\infty \frac{d\varepsilon}{z^{-1}e^{\beta\varepsilon}+1} = \frac{2\pi m}{\beta h^2}\ln(1+z_F)$

Since $N = N_B = N_F$, $-\ln(1-z_B) = \ln(1+z_F) \rightarrow (1-z_B)^{-1} = (1+z_F)$

$\Rightarrow (1+z_F)(1-z_B) = 1 \quad \square$

$\Rightarrow z_B = \frac{z_F}{1+z_F}$ (just simple algebra). (1)

Lemma 2: $f_2(z_F) = g_2\left(\frac{z_F}{1+z_F}\right) + \frac{1}{2}\ln^2(1+z_F)$

Proof: using Appendix E.6, $\frac{\partial}{\partial z} f_2(z) = \frac{1}{z} f_1(z) \Rightarrow f_2(z_F) = \int_0^{z_F} \frac{1}{z} f_1(z)\,dz$

and $f_1(z) = \int_0^\infty \frac{dx}{z^{-1}e^x+1} = \ln(1+z)$

Thus, $f_2(z_F) = \int_0^{z_F} \frac{\ln(1+z)}{z}\,dz$ (note z is just an integration variable).

u-sub

Now, let $y = \frac{z}{1+z} \Rightarrow z = \frac{-y}{y-1}$, also $z+1 = \frac{y}{1-y} + \frac{1-y}{1-y} = \frac{1}{1-y}$

$\Rightarrow dz = \left(\frac{y}{(y-1)^2} - \frac{1}{y-1}\right)dy = \frac{y-y+1}{(y-1)^2}\,dy = \frac{1}{(y-1)^2}\,dy$

$\Rightarrow f_2(z_F) = \int_0^y \frac{\ln(1/(1-y))\,\cancel{(1-y)}}{y\,(y-1)^{\cancel{2}}}\,dy = \int_0^y \frac{\ln(1/(1-y))\,dy}{y(1-y)} = \int_0^y \frac{\ln(1-y)}{y(y-1)}\,dy$

using D.10, $\frac{\partial}{\partial z} g_2(z) = \frac{1}{z} g_1(z) \Rightarrow g_2(z) = -\int \frac{\ln(1-z)}{z}\,dz$

$\Rightarrow g_2(y) = +\int \frac{\ln(1+y/(y-1))}{y}\,\frac{\cancel{(y-1)}}{(y-1)^{\cancel{2}}}\,dy = \int \frac{\ln(1+y)}{y(y-1)}\,dy - \int \frac{\ln(y-1)}{y(y-1)}\,dy$

$\therefore f_2(z_F) = g_2(y_F) - \int_0^{y_F} \frac{\ln(1+y_F)}{y(y_F-1)}\,dy_F = g_2\left(\frac{z_F}{1+z_F}\right) + \frac{1}{2}\left(\ln\left(1-\frac{z_F}{1+z_F}\right)\right)^2$ ← last step w/ Mathematica...

and $\ln\left(1+\frac{-z_F}{1+z_F}\right) = \ln\left(\frac{1}{1+z_F}\right) = -\ln(1+z_F)$

$\Rightarrow f_2(z_F) = g_2\left(\frac{z_F}{1+z_F}\right) + \frac{1}{2}\ln^2(1+z_F) \quad \square \quad \overset{\text{using (1)}}{=} g(z_B) + \frac{1}{2}\ln^2(1+z_F)$ (2)

Lemma 3: $E_F(N,T) = E_B(N,T) + \text{const.}$

Proof: $E = -\frac{\partial}{\partial\beta}\ln Q \qquad \ln Q_B = -\int a(\varepsilon)\ln(1-ze^{-\beta\varepsilon})\,d\varepsilon = \frac{-2\pi m L^2}{h^2}\int \ln(1-ze^{-\beta\varepsilon})\,d\varepsilon$

$\ln Q_B = \frac{+2\pi m L^2}{\beta h^2}\,g_2(z) \Rightarrow -\frac{\partial}{\partial\beta}\ln Q_B = \frac{2\pi m L^2}{\beta^2 h^2}\,\Gamma(2)\,g_2(z) = E_B.$

Similarly, $E_F = \frac{2\pi m L^2}{\beta^2 h^2}\,\Gamma(2)\,f_2(z) = \frac{2\pi m L^2}{\beta^2 h^2}\,\Gamma(2)\left[g_2(z_B) + \frac{1}{2}\ln^2(1+z_F)\right]$

$\Rightarrow E_F = E_B + \frac{2\pi m L^2}{2\beta^2 h^2}\ln^2(1+z_F)$

Thus, we need to show $\frac{\pi m L^2}{\beta^2 h^2} \ln^2(1+z_F) = \text{const.}$

From lemma 1: $N = \frac{2\pi m}{\beta^2 h^2} \ln(1+z_F) \Rightarrow \ln^2(1+z_F) = \frac{N^2 \beta^4 h^4}{4\pi^2 m^2}$

$\Rightarrow E_F = E_B + \text{constant}$. $\square$

Finally, $C_V = \frac{\partial E}{\partial T} \Rightarrow C_{V,F} = \frac{\partial E_F}{\partial T} = C_{V_B} = \frac{\partial E_B}{\partial T} \Rightarrow \boxed{C_{V,F} = C_{V,B}}$

$\frac{20}{20}$