8.10 we begin by finding the density of states.

(a) $\Sigma(p) = \frac{1}{h^n}\int d^n q\, d^n p = \frac{1}{h^n} V(q) V(p) = \frac{V p^n}{h^n} \frac{\pi^{n/2}}{(n/2)!}$ by appendix C.7a

$g(p)dp = \frac{d\Sigma(p)}{dp} dp = \frac{n V p^{n-1}}{h^n} \frac{\pi^{n/2}}{(n/2)!} dp$

Now $\varepsilon = A p^s \Rightarrow p = \left(\frac{\varepsilon}{A}\right)^{1/s}$, so $dp = d\varepsilon \frac{dp}{d\varepsilon} = \frac{1}{s A^{1/s}} \varepsilon^{1/s - 1} d\varepsilon$

and $p^{n-1} = \left(\frac{\varepsilon}{A}\right)^{\frac{n}{s}-\frac{1}{s}}$

Plugging it all back in, $a(\varepsilon)d\varepsilon = \frac{nV}{h^n}\left(\frac{\varepsilon}{A}\right)^{\frac{n}{s}-\frac{1}{s}} \frac{\pi^{n/2}}{(n/2)!} \frac{1}{sA^{1/s}} \varepsilon^{1/s-1} d\varepsilon$

$\Rightarrow a(\varepsilon)d\varepsilon = \frac{nV}{sh^n} \frac{\pi^{n/2}}{(n/2)!} \frac{1}{A^{n/s}} \varepsilon^{\frac{n}{s}-1} d\varepsilon$

Now find the grand partition function

$\ln \mathcal{Q} = \sum_\varepsilon \ln(1 + z e^{-\beta\varepsilon}) \longrightarrow \int_0^\infty \ln(1+ze^{-\beta\varepsilon})\, a(\varepsilon) d\varepsilon$

$\Rightarrow \ln\mathcal{Q} = \frac{nV}{sh^n}\frac{\pi^{n/2}}{(\frac{n}{2})!}\frac{1}{A^{n/s}}\int_0^\infty \ln(1+ze^{-\beta\varepsilon})\,\varepsilon^{\frac{n}{s}-1} d\varepsilon$

Set $u = \ln(1+ze^{-\beta\varepsilon})$, $dv = \varepsilon^{\frac{n}{s}-1}d\varepsilon \Rightarrow \frac{du}{d\varepsilon} = \frac{-\beta z e^{-\beta\varepsilon}}{1+ze^{-\beta\varepsilon}}$, $v = \frac{s}{n}\varepsilon^{n/s}$

$\therefore \ln\mathcal{Q} = \frac{nV}{sh^n}\frac{\pi^{n/s}}{(n/2)!}\frac{1}{A^{n/s}}\left[\underbrace{\frac{s}{n}\varepsilon^{n/s}\ln(1+ze^{-\beta\varepsilon})\Big|_0^\infty}_{0} - \int_0^\infty \frac{s}{n}\varepsilon^{n/s}\frac{(-\beta z e^{-\beta\varepsilon})}{1+ze^{-\beta\varepsilon}} d\varepsilon\right]$

$= \frac{\cancel{n}V}{\cancel{s}h^n}\frac{\pi^{n/s}}{(n/2)!}\frac{1}{A^{n/s}}\left[\int_0^\infty \frac{s}{n}\varepsilon^{n/s}\frac{\beta z e^{-\beta\varepsilon}}{1+ze^{-\beta\varepsilon}}d\varepsilon\right] = \frac{\beta z V}{h^n}\frac{\pi^{n/s}}{(n/2)!}\frac{1}{A^{n/s}}\int_0^\infty \frac{\varepsilon^{n/s}e^{-\beta\varepsilon}}{1+ze^{-\beta\varepsilon}}d\varepsilon$

$= \frac{\beta V}{A^{n/s}h^n}\frac{\pi^{n/s}}{(n/2)!}\int_0^\infty \frac{\varepsilon^{n/s}}{1+z^{-1}e^{\beta\varepsilon}}d\varepsilon$ letting $x \equiv \beta\varepsilon$: $= \frac{1}{\beta^{n/s}}\frac{V}{A^{n/s}h^n}\frac{\pi^{n/s}}{(\frac{n}{2})!}\int_0^\infty \frac{x^{n/s}}{1+z^{-1}e^x}dx$

Using Fermi-Dirac Functions:

$\ln\mathcal{Q} = \frac{V}{(\beta A)^{n/s}h^n}\frac{\pi^{n/s}}{(n/2)!}\Gamma\left(\frac{n}{s}+1\right) f_{\frac{n}{s}+1}(z) = \frac{PV}{kT}$ (†)

To get the equation of state now, let's get $U$:

$U = kT^2\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right) = kT^2\frac{\partial}{\partial T}\ln\mathcal{Q} = kT^2\frac{V}{h^n A^{n/s}}\frac{\pi^{n/s}}{(\frac{n}{2})!}\Gamma\left(\frac{n}{s}+1\right)f_{\frac{n}{s}+1}(z)\frac{\partial}{\partial T}\left[(kT)^{n/s}\right]$

$= k^{n/s+1}\frac{n}{s}T^{\frac{n}{s}+1}\frac{V}{h^nA^{n/s}}\frac{\pi^{n/s}}{(\frac{n}{2})!}\Gamma\left(\frac{n}{s}+1\right)f_{\frac{n}{s}+1}(z) = \frac{n}{s}\underbrace{\frac{V}{(\beta A)^{n/s}h^n}\frac{\pi^{n/s}}{(\frac{n}{2})!}\Gamma\left(\frac{n}{s}+1\right)f_{\frac{n}{s}+1}(z)}\frac{1}{\beta}$

$= \frac{n}{s}\frac{PV}{kT}\cdot\frac{1}{\beta} = \frac{n}{s}PV \Rightarrow \boxed{PV = \frac{s}{n}U}$ □

(b) $C_V = \frac{dU}{dT}\Big|_N$, so let's write $U$ in terms of $N$.

By 8.1.2

$N = \sum_\varepsilon \frac{1}{z^{-1}e^{\beta\varepsilon}+1} \longrightarrow \int_{\varepsilon=0}^\infty a(\varepsilon)d\varepsilon\frac{1}{z^{-1}e^{\beta\varepsilon}+1} = \frac{n}{s}\frac{V}{h^n}\frac{\pi^{n/2}}{(\frac{n}{2})!}\frac{1}{A^{n/s}}\int\frac{\varepsilon^{n/s-1}}{z^{-1}e^{\beta\varepsilon}+1}d\varepsilon$

Let $\mathcal{J} \equiv \frac{\pi^{n/2}}{(\frac{n}{2})!}\frac{V}{h^n}\frac{1}{A^{n/s}}$ and make the substitution $x \equiv \beta\varepsilon$

$\therefore N = \frac{n}{s}\frac{1}{\beta^{n/s}}\mathcal{J}\int_0^\infty\frac{x^{n/s-1}dx}{z^{-1}e^x+1} \Rightarrow N = \frac{n}{s}(kT)^{n/s}\mathcal{J}\,\Gamma\left(\frac{n}{s}\right)f_{\frac{n}{s}}(z)$ (☆)

we found above that $U = \frac{n}{s}(kT)^{n/s}\mathcal{J}\,\Gamma\left(\frac{n}{s}+1\right)f_{\frac{n}{s}+1}(z)\,kT$

$\Rightarrow U = NkT\frac{\Gamma(\frac{n}{s}+1)f_{\frac{n}{s}+1}(z)}{\Gamma(\frac{n}{s})f_{\frac{n}{s}}(z)} = \frac{n}{s}NkT\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)}$

next, let's take the derivative $\frac{C_v}{Nk} = \frac{1}{Nk}\frac{dU}{dT}$

$$\frac{C_v}{Nk} = \frac{n}{s}\frac{d}{dT}\left(T\,\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)}\right) = \frac{n}{s}\left[\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)} + T\frac{d}{dT}\left(\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)}\right)\right]$$

Now, note $\frac{d}{dT} = \frac{dz}{dT}\frac{d}{dz} = -\left(\frac{n}{s}\right)\frac{1}{T}\frac{f_{\frac{n}{s}}(z)}{f_{\frac{n}{s}-1}(z)}\,z\frac{d}{dz}$ by 8.1.9.

Then use E.6: $z\frac{d}{dz}f_\nu(z) = f_{\nu-1}(z)$

$$\therefore \frac{C_v}{Nk} = \frac{n}{s}\left[\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)} - \left(\frac{n}{s}\right)\frac{f_{\frac{n}{s}}(z)}{f_{\frac{n}{s}-1}(z)}\left(\frac{f_{\frac{n}{s}}(z)\,f_{\frac{n}{s}}(z) - f_{\frac{n}{s}+1}(z)\,f_{\frac{n}{s}-1}(z)}{f_{\frac{n}{s}}(z)\,f_{\frac{n}{s}}(z)}\right)\right]$$

$$\frac{C_v}{Nk} = \frac{n}{s}\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)} + \left(\frac{n}{s}\right)^2\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)} - \left(\frac{n}{s}\right)^2\frac{f_{\frac{n}{s}}(z)}{f_{\frac{n}{s}-1}(z)}$$

$$\boxed{\frac{C_v}{Nk} = \frac{n}{s}\left(\frac{n}{s}+1\right)\frac{f_{\frac{n}{s}+1}(z)}{f_{\frac{n}{s}}(z)} - \left(\frac{n}{s}\right)^2\frac{f_{n/s}(z)}{f_{n/s-1}(z)}}$$

d) use equation ⓕ from the previous page to solve for $P(T)$:

$$P = (kT)^{n/s+1}\frac{1}{A^{n/s}h^n}\frac{\pi^{n/2}}{\left(\frac{n}{2}\right)!}\Gamma\left(\frac{n}{s}+1\right)f_{\frac{n}{s}+1}(z)$$

Use equation ⊛ from the previous page to solve for $V(T)$:

$$V = N\frac{s}{n}\frac{\left(\frac{n}{2}\right)!}{\pi^{n/2}}\frac{A^{n/s}h^n}{k^{n/s}\Gamma\left(\frac{n}{s}\right)f_{\frac{n}{s}}(z)}\cdot\frac{1}{T^{n/s}}$$

For an adiabatic process, $z = $ constant. Thus we note that everything besides $P$, $T$, and $V$ in these equations are constants:

$\frac{P}{T^{n/s+1}} = \text{const.}$, $\quad VT^{n/s} = \text{const.} \Rightarrow T = \frac{\text{const}}{V^{s/n}}$

$$\Rightarrow \frac{P}{\left(\frac{\text{const}}{V^{s/n}}\right)^{n/s+1}} = \text{constant} \Rightarrow \boxed{PV^{1+\left(\frac{s}{n}\right)} = \text{constant}}$$

I don't know how to do ⓒ and ⓔ.

c) $C_p = \partial/\partial T\,(U+PV)$, subtract $C_v$ to simplify

e) $T \gg T_F$ or $T \ll T_F$ affects $z$ & $f_\nu(z)$; change to evaluate

(need next order for $T \ll T_F$ expression)

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