6.11 (a) First off, we notice $\int f(\vec{p})\,d\vec{p} = 1 = \frac{Q_1}{Q_1} = \frac{\frac{1}{h^3}\int e^{-\beta\varepsilon(\vec{p})}d\vec{p}\,d\vec{q}}{Q_1}$

$$\Rightarrow \int f(\vec{p})\,d\vec{p} = \int \frac{1}{h^3}\frac{1}{Q_1}\cdot 4\pi V e^{-\beta\varepsilon(p)}\,p^2\,dp$$

comparing arguments

$$\Rightarrow f(\vec{p})\,d\vec{p} = \frac{\frac{V}{h^3}e^{-\beta\varepsilon(p)}4\pi p^2\,dp}{Q_1} \qquad ⭐ \qquad \text{This is key.}$$

$$Q_1 = \frac{1}{h^3}\int e^{-\beta\varepsilon(\vec{p})}d\vec{p}\,d\vec{q} = \frac{V}{h^3}\int e^{-\beta\varepsilon(\vec{p})}d\vec{p} = \frac{V}{h^3}\int e^{-\beta c(p^2+m_0^2c^2)^{1/2}}d\vec{p}$$

convert to spherical:

$$= \frac{V}{h^3}\int e^{-\beta c(p^2+m_0^2c^2)^{1/2}}4\pi^2 p^2\,dp = \frac{4\pi V}{h^3}\int e^{-\beta c^3 m_0\left(1+\frac{p^2}{m_0^2c^2}\right)^{1/2}}p^2\,dp.$$

U-substitution: let $\cosh\theta = \sqrt{1+\frac{p^2}{m_0^2c^2}}$

Now, note $\frac{d\cosh\theta}{dp} = \frac{d}{dp}\sqrt{1+\frac{p^2}{m_0^2c^2}} = \frac{p/m_0^2c^2}{\sqrt{1+\frac{p^2}{m_0^2c^2}}} = \frac{d\cosh\theta}{d\theta}\cdot\frac{d\theta}{dp} = \sinh\theta\frac{d\theta}{dp}$

$$\Rightarrow \sinh\theta\,d\theta = \frac{p/m_0^2c^2}{\sqrt{1+p^2/m_0^2c^2}} = \frac{1}{\cosh\theta\, m_0^2c^2}p\,dp \Rightarrow p\,dp = m_0^2c^2\cosh\theta\sinh\theta\,d\theta.$$

Also note that $\cosh^2\theta - 1 = \sinh^2\theta = \frac{p^2}{m_0^2c^2} \Rightarrow p = m_0c\sinh\theta \Rightarrow \frac{p}{m_0c\sinh\theta} = 1$

Multiply by 1: $\left(\frac{p}{m_0c\sinh\theta}\right)p\,dp = m_0^2c^2\cosh\theta\sinh\theta\,d\theta$

$$\Longrightarrow p^2\,dp = (m_0c)^3\cosh\theta\sinh^2\theta\,d\theta$$

Rearrange to make it look like modified Bessel functions.

$$\cosh\theta\sinh^2\theta = \left(\frac{e^\theta+e^{-\theta}}{2}\right)\left(\frac{e^\theta-e^{-\theta}}{2}\right)^2 = \frac{1}{8}\left(e^{-3\theta}-e^{-\theta}-e^{\theta}+e^{3\theta}\right) = \frac{1}{4}\left(\cosh(3\theta)-\cosh\theta\right)$$

$$\Longrightarrow p^2\,dp = \frac{(m_0c)^3}{4}\left(\cosh(3\theta)-\cosh(\theta)\right)d\theta$$

Plug into Q_1: $Q_1 = \frac{4\pi V}{h^3}\int e^{-\beta c^3 m_0\cosh\theta}\left(\frac{(m_0c)^3}{4}[\cosh(3\theta)-\cosh(\theta)]\right)d\theta$

$$\Rightarrow Q_1 = \frac{\pi V(m_0c)^3}{h^3}\left[\int e^{-\beta c^3 m_0\cosh\theta}\cosh(3\theta)\,d\theta - \int e^{-\beta c^3 m_0\cosh\theta}\cosh(\theta)\right]d\theta$$

From Schwinger Electrodynamics 18.71, $K_\nu(x) = \int_0^\infty \cosh(\nu\theta)e^{-x\cosh(\theta)}d\theta$

$$\Rightarrow Q_1 = \frac{\pi V(m_0c)^3}{h^3}\left(K_3(\beta m_0c^2) - K_1(\beta m_0c^2)\right)$$

Mathematica shows us that $K_3(x) - K_1(x) = \frac{4}{x}K_2(x)$ (output attached).

$$\Rightarrow Q_1 = \frac{4\pi V m_0^2c}{h^3\beta}K_2(\beta m_0c^2) \quad \text{from ⭐} \quad f(\vec{p})\,d\vec{p} = \frac{\cancel{\frac{V}{h^3}}e^{-\beta c(p^2+m_0^2c^2)^{1/2}}\,4\pi p^2\,dp}{\cancel{\frac{4\pi V}{h^3}}\frac{m_0^2c}{\beta}K_2(\beta m_0c^2)} \rightarrow$$

$$\Rightarrow \boxed{f(\vec{p})\,d\vec{p} = \frac{\beta e^{-\beta c(p^2+m_0^2c^2)^{1/2}}\, p^2\,dp}{m_0^2 c\, K_2(\beta m_0 c^2)}}$$

(b) $kT \ll m_0c^2$

Binomial Aprox.

$$\sqrt{p^2+m_0^2c^2} = m_0 c\sqrt{1+\frac{p^2}{m_0^2c^2}} \simeq m_0c\left(1+\frac{p^2}{2m_0^2c^2}\right) = m_0c + \frac{p^2}{2m_0c}$$

$$\Rightarrow -\beta c(p^2+m_0^2c^2)^{1/2} \simeq -\beta m_0c^2 - \frac{\beta p^2}{2m_0} = \frac{-m_0c^2}{kT} - \frac{p^2}{2kTm_0}$$

$$\Rightarrow e^{-\beta c(p^2+m_0^2c^2)^{1/2}} \simeq \underbrace{\cancel{e^{-\frac{m_0c^2}{kT}}}}_{1} \cdot e^{-p^2\beta/2m_0}$$

(in the nonrel. limit)

$$\Rightarrow f(\vec{p})\,d\vec{p} = \frac{\frac{4\pi V}{h^3} e^{-\beta p^2/2m_0}\, p^2\,dp}{Q} \quad \text{using } \circledast$$

now, $Q_1 = \frac{1}{h^3}\int_{-\infty}^{\infty} e^{-\beta p^2/2m_0}\,dp^3 = \frac{V}{h^3}\left(\sqrt{\frac{2\pi m}{\beta}}\right)^3$

$$\Rightarrow f(\vec{p})\,d\vec{p} = \frac{\frac{4\pi V}{h^3} e^{-\beta p^2/2m_0}\, p^2\,dp}{\frac{V}{h^3}\left(\sqrt{\frac{2\pi m_0}{\beta}}\right)^3} = \boxed{\left(\frac{\beta}{2\pi m_0}\right)^{3/2} e^{-\beta p^2/2m_0}\, 4\pi p^2\,dp}$$

$kT \gg m_0c^2$ in this limit, $\varepsilon \simeq pc$, so $f(\vec{p})\,d\vec{p} = \frac{\frac{V}{h^3} e^{-\beta pc}\, 4\pi p^2\,dp}{Q_1}$

$$Q_1 = \frac{1}{h^3}\int_{-\infty}^{\infty} e^{-\beta pc}\,d\vec{p}\,d\vec{q} = \frac{4\pi V}{h^3}\left(\int_{-\infty}^{\infty} e^{-\beta pc}\,p^2\,dp\right)^3 = \frac{8\pi}{(\beta c)^3}\,\frac{V}{h^3}$$

$$\Rightarrow f(\vec{p})\,d\vec{p} = \frac{\frac{V}{h^3} e^{-\beta pc}\, 4\pi p^2\,dp}{\frac{8\pi}{(\beta c)^3}\,\frac{V}{h^3}} = \boxed{\frac{(\beta c)^3}{8\pi} e^{-\beta pc}\,(4\pi p^2\,dp)}$$

(c) We use equation $\circledast$ to find $\langle pu\rangle$. (I have multiplied both sides by Q_1 b/c I'm running low on fractions...)

$$Q_1\langle pu\rangle = \frac{1}{h^3}\int d\vec{p}\,d\vec{q}\,(pu)\,e^{-\beta\varepsilon(\vec{p})}$$

Now, since $\varepsilon = \frac{p^2}{2m} \Rightarrow \frac{d}{dp}\varepsilon = \frac{2p}{2m} = \frac{mu}{u} = u \Rightarrow Q_1\langle pu\rangle = \frac{1}{h^3}\int \cancel{d\vec{p}}\,d\vec{q}\;p\frac{d\varepsilon}{\cancel{dp}}\,e^{-\beta\varepsilon(p)}$

$$Q_1\cdot\langle pu\rangle = \frac{4\pi V}{h^3}\int p^3\,d\varepsilon\,e^{-\beta\varepsilon(p)}$$

next, $\varepsilon^2 = (pc)^2 + (mc^2)^2 \Rightarrow p^2 = \frac{\varepsilon^2}{c^2} - mc^2 \Rightarrow p^3 = \left(\left(\frac{\varepsilon}{c}\right)^2 - m^2c^2\right)^{3/2}$

$$Q_1\cdot\langle pu\rangle = (mc)\frac{4\pi V}{h^3}\int\left[\frac{\varepsilon^2}{m^2c^4} - 1\right]^{3/2} e^{-\beta\varepsilon(p)}\,d\varepsilon \qquad \text{Let } x = \frac{\varepsilon}{mc^2}$$

$$= (m^4c^5)\,\frac{4\pi V}{h^3}\int e^{-\beta mc^2 x}\,[x^2-1]^{3/2}\,dx = K_2(\beta mc^2)\,\frac{(\frac{3}{2})!}{\sqrt{\pi}}\left(\frac{\beta mc^2}{2}\right)^{-2}(m^4c^5)\frac{4\pi V}{h^3}$$

Because $\int_1^{\infty} e^{-zx}(x^2-1)^{n-1/2}\,dx = K_n(z)\,\frac{(n-\frac{1}{2})!}{\sqrt{\pi}}\left(\frac{1}{2}z\right)^{-n}$

6.11 cont

Therefore $\langle pu \rangle = \dfrac{\cancel{K_2(\beta m c^2)}\, \frac{\overset{\to 3\sqrt{\pi}/4}{\left(\frac{3}{2}\right)!}}{\sqrt{\pi}} \left(\frac{\beta m c^2}{2}\right)^{-2} m^4 c^5\, \cancel{\frac{4\pi V}{h^3}}}{\cancel{\frac{4\pi V m^2 c}{h^3 \beta}}\, \cancel{K_2(\beta m c^2)}} = \dfrac{3}{\beta} = \boxed{3kT}$

$\frac{10}{10}$

Wow - I almost lost faith there...
(see simplification in Mathematica attachment).

Modified Bessel Function difference identity

In[381]:=
```
BesselK[3, θ] - BesselK[1, θ] // FullSimplify
```

Out[381]= $\dfrac{4\,\mathrm{BesselK}[2, \theta]}{\theta}$

The big simplification from the end of part c.

In[402]:= $\dfrac{\frac{\left(\frac{3}{2}\right)!}{\pi^{1/2}} \left(\beta\, \frac{m\, c^2}{2}\right)^{-2} m^4\, c^5}{\frac{m^2\, c}{\beta}}$

Out[402]= $\dfrac{3}{\beta}$