4.4 The probability of finding a system in the Grand Canonical Ensemble to have Energy E_s and particle number N_r is given by 4.2.9:

$$P(N_r, E_s) = \frac{e^{-\alpha N_r} e^{-\beta E_s}}{\sum_{r,s} e^{-\alpha N_r} e^{-\beta E_s}} \quad \text{where } \alpha = \frac{-\mu}{kT}, \text{ and } \beta = \frac{1}{kT}.$$

Thus, the probability of finding a system to have exactly N particles can be found by letting $N_r \to N$, and summing over E_s:

$$p(N) = \sum_s \frac{e^{-\alpha N} e^{-\beta E_s}}{\sum_{r,s} e^{-\alpha N_r} e^{-\beta E_s}} = \sum_s \frac{z^N e^{-\beta E_s}}{\sum_r \left(z^N \sum_s e^{-E_s/kT}\right)} \quad \text{by 4.3.11 } (e^{-\alpha} = z)$$

by 3.3.8, $Q_N(V,T) = \sum_s e^{-E_s/kT}$, so $P(N) = \dfrac{z^N Q_N(V,T)}{\sum_r z^{N_r} Q_{N_r}(V,T)}$

by 4.3.15, the denominator is simply the grand partition function, so we end up with

$$\boxed{P(N) = \frac{z^N Q_N(V,T)}{\mathcal{Q}(z,V,T)}}$$

For a classical ideal gas, $Q_N(V,T) = \frac{(Vf(T))^N}{N!}$ using 4.4.1, and 4.4.2.
and by 4.4.3, $\mathcal{Q}(z,V,T) = e^{zVf(T)}$

$$P(N) = \frac{z^N (Vf(T))^N}{N!\, e^{zVf(T)}} = \frac{z^N e^{-zVf(T)} (Vf(T))^N}{N!} = \boxed{\frac{[zVf(T)]^N e^{-[zVf(T)]}}{N!}}$$

This is a poisson distribution.
In general, $P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$. Here we let $\lambda \to zVf(T)$, and $k \to N$, and we recover the boxed expression.

For an unbiased estimator like ΔN, the root mean square deviation $\left(\sqrt{\overline{(\Delta N)^2}}\right)$ is the same as the square root of variance $\sqrt{\mathrm{var}(\Delta N)}$ (also called "Standard Error").
For a poisson distribution, the variance is simply λ.

$$\therefore \quad \sqrt{\mathrm{Var}(\Delta N)} = \sqrt{\lambda} = \boxed{\sqrt{zVf(T)}}$$

General formula
Meanwhile, by 4.5.3, $\overline{(\Delta N)^2} = kT\left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V} \Rightarrow \sqrt{\overline{(\Delta N)^2}} = \sqrt{kT\left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V}}$

We know for a classical ideal gas, $\bar{N} = zVf(T)$ (by 4.4.6)

$$\sqrt{\overline{(\Delta N)^2}} = \sqrt{kT\left(\frac{\partial}{\partial \mu} zVf(T)\right)} = \sqrt{kT\left(\frac{\partial}{\partial \mu} e^{\mu/kT}\right) Vf(T)} = \sqrt{\frac{kT}{kT} zVf(T)} = \boxed{\sqrt{zVf(T)}}$$

These two methods agree!!!

19/20

Note: if we assume the ideal gas to be monatomic, $f(T) = (2\pi mkT)^{3/2}/h^3$