2-9

(a) $\underbrace{\int\cdots\int}_{0\le\sum_{i=1}^{3N}|x_i|\le R} \prod_{i=1}^{3N} dx_i \equiv V$ and let $\sum_{i=1}^{3N}|x_i| = R \Rightarrow \prod_{i=1}^{3N} dx_i = dR$

Thus, per C.3 $dV = (3N)C_{3N}R^{3N-1}dR \Rightarrow \int_{-\infty}^{\infty} dV = 3NC_{3N}\underbrace{\int_0^{\infty} dR\, R^{3N-1}}$

The radial Part

→ Integration trick! multiply both sides by e^{-R}.

$$\Rightarrow \int_{-\infty}^{\infty} dVe^{-R} = 3NC_{3N}\int_0^{\infty} dRe^{-R}R^{3N-1}$$

→ To evaluate the integral, we once again note that $\int_0^{\infty} e^{-x}x^k dx = k!$

$$\Rightarrow \int_{-\infty}^{\infty} dVe^{-R} = 3NC_{3N}(3N-1)!$$

$$\int_0^{\infty}\prod_{i=1}^{3N} dx_i\, e^{-\sum_{i=1}^{3N}|x_i|} = 3NC_{3N}(3N-1)!$$

$$\underbrace{\left[\int_0^{\infty} dx_i e^{-|x_i|}\right]^{3N}}_{= 2^{3N}} = 3NC_{3N}(3N-1)! \Rightarrow 8\pi = 3NC_{3N}(3N-1)!$$

$$\therefore C_{3N} = \frac{2^{3N}}{(3N)!} \quad \text{Thus } V_{3N} = \frac{(2R)^{3N}}{(3N)!} \text{ by C.2}$$

Now, $p = \hbar k = \hbar\frac{n\pi}{D} \Rightarrow \varepsilon = \frac{c\hbar n\pi}{D} \Rightarrow E = \frac{c\hbar\pi}{D}\sum_{i=1}^{3N} n \Rightarrow R = \frac{ED}{c\hbar\pi}$

The "volume in phase space" is $\omega = \underbrace{\int\cdots\int}_{0\le\sum_{i=1}^{3N}|p_i|\le E/c} d^{3N}p \cdot \underbrace{\int\cdots\int d^{3N}q}_{\to D^{3N}}$

$$\omega = V(E/c)\cdot D^{3N} = \boxed{\frac{(2ED)^{3N}}{c^{3N}(3N)!} = \omega}$$

where the 1-D box is of length D.

Now we look for Ω: the number of ways of distributing E amongst the particles. As explained in 1-8 and 2-7,

$$\Omega = \frac{(R+N-1)!}{R!(N-1)!} \overset{\text{see 2-7}}{\Longrightarrow} \Omega \simeq \frac{(ED)^{3N}}{c^{3N}(3N!)}\frac{1}{(\hbar\pi)^{3N}} = \boxed{\frac{(ED)^{3N}}{c^{3N}(3N!)}\left(\frac{2}{h}\right)^{3N} = \Omega}$$

$$\Omega = \frac{\omega}{\omega_0} \Rightarrow \omega_0 = \frac{\Omega}{\omega} = \frac{\frac{(2ED)^{3N}}{c^{3N}(3N)!}}{\frac{(2ED)^{3N}}{c^{3N}(3N)!}\frac{1}{h^{3N}}} = \boxed{h^{3N}}$$

(b) Now I use the Ω calculated in (a) to find $\frac{C_P}{C_V}$, etc.: $\Omega = \frac{(2ED)^{3N}}{(hc)^{3N}(3N)!}$

$$S = k\ln\Omega = k\ln\left[\frac{(2ED)^{3N}}{(hc)^{3N}(3N!)}\right]$$

$$\Rightarrow T = \frac{1}{\left(\frac{\partial S}{\partial E}\right)_{N,V}} = \boxed{\frac{E}{3kN} = T} \leftarrow \text{(see mathematica)}$$

Setting $D^{3N} = V^N$ $\{V^N = D^N;$ 1-D particles$\}$

$V^N = D^N$

$$P = \frac{\left(\frac{\partial S}{\partial V}\right)_{N,E}}{\left(\frac{\partial S}{\partial E}\right)_{N,V}} = \boxed{\frac{E}{3V} = P} \Rightarrow \boxed{PV = \frac{E}{3}}$$

We see from the expression for T that $E = 3kNT$,

so $C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \boxed{3Nk = C_V}$

and $C_P = \left(\frac{\partial(E+PV)}{\partial T}\right)_{N,P} = \left(\frac{\partial(E+\frac{E}{3})}{\partial T}\right)_{N,P} = \frac{\partial\, 4NkT}{\partial T} = \boxed{4Nk = C_P}$

$$\Rightarrow \boxed{C_P/C_V = \frac{4}{3}}$$

Also $\mu = \left(\frac{\partial E}{\partial N}\right)_{V,S} = \boxed{3kT = \mu}$

Assuming we set $D^{3N} = V^N$, it all works out the same!

8/10

```
S[N_, V_, E_] := k Log[ V^N (2 E)^(3 N) / ((h c)^(3 N) (3 N) !) ]

1 / D[S[N, V, E], E]   (*Temperature*)

E/(3 k N)

D[S[N, V, E], V] / D[S[N, V, E], E]   (*Pressure*)

E/(3 V)
```