Goldstein 9-6

(a) A transformation is canonical when $M^T J M = J$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (from 8.38a), and M is the Jacobian Matrix. By 9.51 M has elements $M_{ij} = \dfrac{\partial \zeta_i}{\partial \eta_j}$ ← transformed ← orig. coords.

Thus, $M = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}$

Given $Q = \log(1 + q^{1/2}\cos p)$, $P = 2(1+q^{1/2}\cos p)q^{1/2}\sin p$.

Taking the partials:

$$M = \begin{bmatrix} \frac{1}{2}\left(\frac{1}{q + \sqrt{q}\sec(p)}\right) & \frac{-\sqrt{q}\sin(p)}{1+\sqrt{q}\cos(p)} \\ \left(\frac{1}{\sqrt{q}} + 2\cos(p)\right)\sin(p) & 2\left(\sqrt{q}\cos(p) + q\cos(2p)\right) \end{bmatrix}$$

Now, all that's left is to do $M^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M$, and see if it equals $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Leaving that to Mathematica - see attached - we get

$M^T J M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ □ Thus, so long as q, p are canonical, Q and P are too.

(b) Type 3 generating functions have the form $q = \dfrac{-\partial F_3(p,Q,t)}{\partial p}$, $P = \dfrac{-\partial F_3(p,Q,t)}{\partial Q}$

We will show that $F_3(p,Q,t) = -(e^Q - 1)^2 \tan(p)$ gives us the above transformations.

Take those derivatives:

(1) $q = \dfrac{-\partial F_3}{\partial p} = (e^Q - 1)^2 \sec^2(p)$

(2) $P = \dfrac{-\partial F_3}{\partial Q} = \tan(p)\left(\dfrac{\partial}{\partial Q}(e^Q-1)(e^Q-1)\right) = \tan(p)\left(\dfrac{\partial}{\partial Q}(e^{2Q} - 2e^Q + 1)\right)$

$= \tan(p)(2e^{2Q} - 2e^Q) = 2\tan(p)e^Q(e^Q - 1)$

Solve (1) for Q: $\dfrac{q^{1/2}}{\sec(p)} + 1 = e^Q \Rightarrow \boxed{Q = \log(1 + \cos(p)q^{1/2})}$

Now plug that into (2):

$P = 2\tan(p)\, e^{\log(1+\cos(p)q^{1/2})}\left(e^{\log(1+\cos(p)q^{1/2})} - 1\right)$

$= 2\tan(p)(1 + \cos(p)q^{1/2})(1 + \cos(p)q^{1/2} - 1) = 2\sin(p)q^{1/2}(1 + \cos(p)q^{1/2})$

$\boxed{P = 2(1+q^{1/2}\cos p)q^{1/2}\sin p}$ So, it worked!

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Q[q_, p_] = Log[1 + q^(1/2) Cos[p]];
P[q_, p_] = 2 (1 + q^(1/2) Cos[p]) q^(1/2) Sin[p];

M = {{D[Q[q, p], q], D[Q[q, p], p]}, {D[P[q, p], q], D[P[q, p], p]}} // FullSimplify;

Transpose[M].{{0, 1}, {-1, 0}}.M // FullSimplify // MatrixForm
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$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$