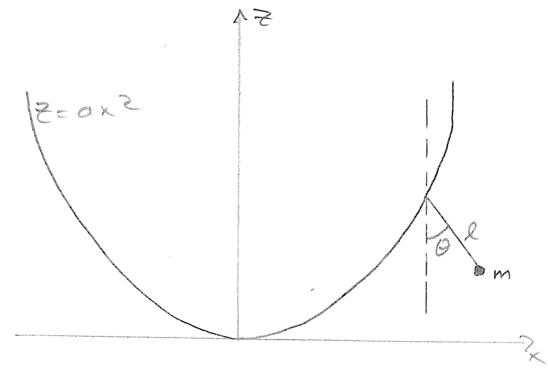
5 Goldstein 8-19

Given the pendulum sliding on a parabola, I will use x and θ as my gen. coordinates.



The first step is to find T and U

→ If (x, z) is the pendulum's pivot point,

then $x' = x + l\sin\theta$, $z' = z - l\cos\theta$
$z' = ax^2 - l\cos\theta$

and $T = \frac{1}{2}m(\dot{x}'^2 + \dot{z}'^2)$
carry this out in Mathematica to get:

$$T = \frac{1}{2}m\left[(\dot{x} + \dot{\theta}l\cos\theta)^2 + (2ax\dot{x} + \dot{\theta}l\sin\theta)^2\right]$$

$$T = \frac{1}{2}m\left[\dot{x}^2 + 4a^2x^2\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos(\theta) + 4al\dot{x}\dot{\theta}x\sin\theta + l^2\dot{\theta}^2\right]$$

$$U = mgz' = mgl[ax^2 - \cos\theta]$$

$$L = \frac{1}{2}m\dot{x}^2 + 2ma^2x^2\dot{x}^2 + m\dot{x}\dot{\theta}l\cos\theta + 2mal\dot{x}\dot{\theta}x\sin\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mglax^2 + mgl\cos\theta$$

Now the conjugate Momenta:

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + 4ma^2x^2\dot{x} + m\dot{\theta}l\cos\theta + 2mal\dot{\theta}\sin\theta$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\dot{x}l\cos\theta + 2mal\dot{x}x\sin\theta + ml^2\dot{\theta}$$

Solve the momenta for $\dot{x}$ and $\dot{\theta}$ (see mathematica for the simplification).

$$\boxed{\dot{x} = \frac{lP_x - P_\theta(\cos\theta + 2a\sin\theta\, x)}{ml(\sin\theta - 2a\cos\theta\, x)^2} \qquad \dot{\theta} = \frac{P_\theta + 4a^2P_\theta x^2 - lP_x(\cos\theta + 2a\sin\theta\, x)}{ml^2(\sin\theta - 2a\cos\theta\, x)^2}}$$

Now write down the Hamiltonian

$$H = \dot{\theta}P_\theta + \dot{x}P_x - L = \frac{1}{2}m\left(-2gl\cos\theta + \dot{x}^2 + 2ax^2(gl + 2a\dot{x}^2) + 2l(\cos\theta + 2a\sin\theta\, x)\dot{x}\dot{\theta} + l^2\dot{\theta}^2\right)$$

Plug in $\dot{x}$ and $\dot{\theta}$ from above and get $H(x, \theta, P_x, P_\theta)$
See Mathematica — no way I'm copying all that!

Now take Derivatives to get final two eqs. of motion.

$$\boxed{\begin{aligned} \dot{P}_x &= \frac{-2l^2P_x - 2lP_\theta(\cos\theta + 2a\sin\theta\, x)}{2ml^2(\sin\theta - 2a\cos\theta\, x)^2} \\ \dot{P}_\theta &= \frac{-2lP_x(\cos\theta + 2a\sin\theta\, x) + 2P_\theta(1 + 4a^2x^2)}{2ml^2(\sin\theta - 2a\cos(\theta)\, x)^2} \end{aligned}}$$

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Take the derivatives to get the Kinetic Energy T
I put a “p” after variables to denote them being primed.

```
xp[t_] = x[t] + ι Sin[θ[t]];
zp[t_] = a (x[t])^2 - ι Cos[θ[t]];
(D[xp[t], t])^2 + (D[zp[t], t])^2
```

$$(x'[t] + \iota\,\mathrm{Cos}[\theta[t]]\,\theta'[t])^2 + \left(2\,a\,x[t]\,x'[t] + \iota\,\mathrm{Sin}[\theta[t]]\,\theta'[t]\right)^2$$

Here is the Lagrangian

```
L[x_, θ_, t_] = 1/2 m (x'[t])^2 + 2 m a^2 x[t]^2 (x'[t])^2 + m (x'[t]) (θ'[t]) ι Cos[θ[t]] +
    2 m a ι x'[t] θ'[t] x[t] Sin[θ[t]] + 1/2 m ι^2 (θ'[t])^2 - m g ι a (x[t])^2 + m g ι Cos[θ[t]];
```

And the Conjugate Momenta

```
Px[x_, θ_, t_] =
  m x'[t] + 4 m a^2 (x[t])^2 x'[t] + m θ'[t] ι Cos[θ[t]] + 2 m a ι θ'[t] x[t] Sin[θ[t]];
Pθ[x_, θ_, t_] = m x'[t] ι Cos[θ[t]] + 2 m a ι x'[t] x[t] Sin[θ[t]] + m ι^2 (θ'[t]);
```

Solve for x dot and θ dot in terms of the conjugate momenta

```
Solve[{Px[t] == Px[x, θ, t], Pθ[t] == Pθ[x, θ, t]}, {x'[t], θ'[t]}] // FullSimplify
```

$$\left\{\left\{x'[t] \to \frac{\iota\,\mathrm{Px}[t] - \mathrm{P}\theta[t]\,\left(\mathrm{Cos}[\theta[t]] + 2\,a\,\mathrm{Sin}[\theta[t]]\,x[t]\right)}{m\,\iota\,\left(\mathrm{Sin}[\theta[t]] - 2\,a\,\mathrm{Cos}[\theta[t]]\,x[t]\right)^2},\right.\right.$$
$$\left.\left.\theta'[t] \to \left(\mathrm{P}\theta[t] + 4\,a^2\,\mathrm{P}\theta[t]\,x[t]^2 - \iota\,\mathrm{Px}[t]\,\left(\mathrm{Cos}[\theta[t]] + 2\,a\,\mathrm{Sin}[\theta[t]]\,x[t]\right)\right) \Big/ \left(m\,\iota^2\,\left(\mathrm{Sin}[\theta[t]] - 2\,a\,\mathrm{Cos}[\theta[t]]\,x[t]\right)^2\right)\right\}\right\}$$

First two equations of motion. Define new variables for x dot and p dot to make it easy to plug them into H.

```
xdot = (ι Px[t] - Pθ[t] (Cos[θ[t]] + 2 a Sin[θ[t]] x[t])) / (m ι (Sin[θ[t]] - 2 a Cos[θ[t]] x[t])^2);
θdot = (Pθ[t] + 4 a^2 Pθ[t] x[t]^2 - ι Px[t] (Cos[θ[t]] + 2 a Sin[θ[t]] x[t])) /
    (m ι^2 (Sin[θ[t]] - 2 a Cos[θ[t]] x[t])^2);
```

Now here is the hamiltonian H

```
H[x_, θ_, t_] = θ'[t] Pθ[x, θ, t] + x'[t] Px[x, θ, t] - L[x, θ, t] // FullSimplify
```

$$\frac{1}{2}\,m\,\left(-2\,g\,\iota\,\mathrm{Cos}[\theta[t]] + x'[t]^2 + 2\,a\,x[t]^2\,\left(g\,\iota + 2\,a\,x'[t]^2\right) + 2\,\iota\,\left(\mathrm{Cos}[\theta[t]] + 2\,a\,\mathrm{Sin}[\theta[t]]\,x[t]\right)\,x'[t]\,\theta'[t] + \iota^2\,\theta'[t]^2\right)$$

And plug in for xdot and θdot

```
H[x_, θ_, Pθ_, Px_, t_] = 1/2 m (-2 g ι Cos[θ[t]] + xdot^2 + 2 a x[t]^2 (g ι + 2 a xdot^2) +
      2 ι (Cos[θ[t]] + 2 a Sin[θ[t]] x[t]) xdot θdot + ι^2 θdot^2) // FullSimplify
```

$$\left(\iota^2\,\mathrm{Px}[t]^2 - 2\,\iota\,\mathrm{Px}[t]\,\mathrm{P}\theta[t]\,\left(\mathrm{Cos}[\theta[t]] + 2\,a\,\mathrm{Sin}[\theta[t]]\,x[t]\right) + 2\,g\,m^2\,\iota^3\,\left(\mathrm{Sin}[\theta[t]] - 2\,a\,\mathrm{Cos}[\theta[t]]\,x[t]\right)^2\,\left(-\mathrm{Cos}[\theta[t]] + a\,x[t]^2\right) + \mathrm{P}\theta[t]^2\,\left(1 + 4\,a^2\,x[t]^2\right)\right) \Big/ \left(2\,m\,\iota^2\,\left(\mathrm{Sin}[\theta[t]] - 2\,a\,\mathrm{Cos}[\theta[t]]\,x[t]\right)^2\right)$$

Finally we differentiate to get the **final two equations of motion**

```
-D[H[x, θ, Pθ, Px, t], Px[t]]
-D[H[x, θ, Pθ, Px, t], Pθ[t]]
```

$$-\frac{2\,\iota^2\,\mathrm{Px}[t] - 2\,\iota\,\mathrm{P}\theta[t]\,\left(\mathrm{Cos}[\theta[t]] + 2\,a\,\mathrm{Sin}[\theta[t]]\,x[t]\right)}{2\,m\,\iota^2\,\left(\mathrm{Sin}[\theta[t]] - 2\,a\,\mathrm{Cos}[\theta[t]]\,x[t]\right)^2}$$

$$-\left(\left(-2\,\iota\,\mathrm{Px}[t]\,\left(\mathrm{Cos}[\theta[t]] + 2\,a\,\mathrm{Sin}[\theta[t]]\,x[t]\right) + 2\,\mathrm{P}\theta[t]\,\left(1 + 4\,a^2\,x[t]^2\right)\right) \Big/ \left(2\,m\,\iota^2\,\left(\mathrm{Sin}[\theta[t]] - 2\,a\,\mathrm{Cos}[\theta[t]]\,x[t]\right)^2\right)\right)$$