Goldstein 6-18

The Kinetic Energy is simple $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ where I've assumed the mass of the particle is m.

This is a harmonic oscillator, so we have a component of the potential given by $\frac{1}{2}k(x^2+y^2+z^2)$.

The E-field contribution to the potential energy is $q\Phi$

The B-field contribution is $-q\vec{A}\cdot\vec{v}$

→ Let $\vec{E} = E\hat{x}$, $\vec{B} = B\hat{y}$ (they are "crossed").

Since $\vec{E} = -\vec{\nabla}\Phi$, $\Phi = -Ex$ works

And, since $\vec{B} = \vec{\nabla}\times\vec{A}$, we can choose $\vec{A} = Bz\hat{x}$ (among other things).

→ Thus, $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}k(x^2+y^2+z^2) + qEx + qBz\dot{x}$

(where motion must be in $\dot{x}$ direction for the final term to take effect.

Find Equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 = m\ddot{x} + qB\dot{z} + kx - qE$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 = m\ddot{y} + ky \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 = m\ddot{z} + kz - qB\dot{x}$$

↳ $\boxed{y(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}} \quad \leftarrow \text{first normal mode}}$

To solve the other two, we've gotta get a little creative. Lets shift our $\ddot{x}$ equation to where the particle is in equilibrium with contributions from $\vec{E}$ and the spring canceling. $x \to \frac{qE}{k} + x$

Then we get $m\ddot{x} + qB\dot{z} + kx = 0$, $m\ddot{z} - qB\dot{x} + kz = 0$ (nice and symmetric).

Guess Solutions of the form:

$x = x_0 e^{-i\omega t}$, $z = z_0 e^{-i\omega t}$ $\Rightarrow$ $\ddot{x} = -\omega^2 x_0 e^{-i\omega t}$ $\quad \ddot{z} = -\omega^2 z_0 e^{-i\omega t}$

Plug them in. $\Rightarrow$ $\dot{x} = -i\omega x_0 e^{-i\omega t}$ $\quad \dot{z} = -i\omega z_0 e^{-i\omega t}$

$$-m\omega^2 x_0 - iqB\omega z_0 + kx_0 = 0, \qquad -m\omega^2 z_0 + qB\omega x_0 + kz_0 = 0$$

$$(k - m\omega^2)x_0 - iqB\omega z_0 = 0 \qquad (k - m\omega^2)z_0 + iqB\omega x_0 = 0$$

$$\begin{cases}(k-m\omega^2)x_0 = iqB\omega z_0 \\ (k-m\omega^2)z_0 = -iqB\omega x_0\end{cases} \Rightarrow (k-m\omega^2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_0\\ z_0\end{bmatrix} = iqB\omega\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_0\\ z_0\end{bmatrix}$$

can be rewritten as $(k-m\omega^2)\hat{T}\vec{x} = iqB\omega\,\hat{U}\vec{x} \Rightarrow \hat{T}^{-1}\hat{U}\vec{x} = \frac{k-m\omega^2}{iqB\omega}\vec{x}$

Set $\hat{M} = \hat{T}^{-1}\hat{U}$, and $\lambda = \frac{k-m\omega^2}{iqB\omega}$, Then we have $\hat{M}\vec{x} = \lambda\vec{x}$

we get $\lambda_1 = i$, $\lambda_2 = -i$ $\Rightarrow$ $\boxed{\omega_1 = \frac{Bq - \sqrt{4km + B^2q^2}}{2m}, \quad \omega_2 = \frac{Bq + \sqrt{4km + B^2q^2}}{2m}}$

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Strong field approx: $B^2q^2 \gg 4km$. Then $\omega \to \frac{Bq + \sqrt{B^2q^2}}{2m}$

$$\boxed{\omega_{strong} \approx \frac{Bq}{m}}$$

[The so-called "cyclotron frequency" for charge in B-fields. We Don't even notice the harmonic oscillator anymore.]

Weak field approx: $4km \gg B^2q^2$

$$\omega \approx \frac{\pm\sqrt{4km}}{2m} = \boxed{\pm\sqrt{\frac{k}{m}} \approx \omega_{weak}}$$

When the fields shrink very low, we recover the solution for a simple harmonic oscillator as expected.

In any case, ω_0 is unaffected.

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