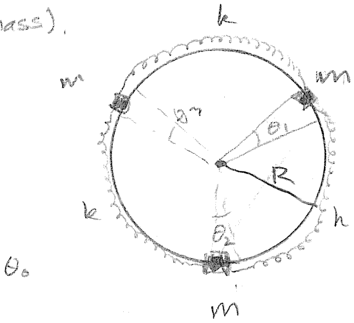
Goldstein 6-10

(a) with the setup shown @ right, the 3 gen. coords are $\theta_1, \theta_2, \theta_3$. (one per mass).



Get Lagrange's Eqs. of Motion

$$T = \tfrac{1}{2} m R^2 \dot{\theta}_1^2 + \tfrac{1}{2} m R^2 \dot{\theta}_2^2 + \tfrac{1}{2} m R^2 \dot{\theta}_3^2$$

$U = \frac{1}{2} k \Delta x_1^2 + \frac{1}{2} k \Delta x_2^2 + \frac{1}{2} k \Delta x_3^2$ and Δx_i is the differ in arc length from the equilibrium position. $R\theta - R\theta_0$

$$U = \tfrac{1}{2} k (R\theta_3 - R\theta_1)^2 + \tfrac{1}{2} k (R\theta_1 - R\theta_2)^2 + \tfrac{1}{2} k (R\theta_2 - R_3)^2$$

$$L = T - U = \tfrac{1}{2} m R^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - \tfrac{1}{2} k \left[(R\theta_3 - R\theta_1)^2 + (R\theta_1 - R\theta_2)^2 + (R\theta_2 - R\theta_3)^2 \right]$$

(1) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 = \ddot{\theta}_1 m R^2 - \tfrac{1}{2} k \left(2R (R\theta_1 - R\theta_2) - 2R (R\theta_3 - R\theta_1)\right)$$

(2) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 = \ddot{\theta}_2 m R^2 - \tfrac{1}{2} k \left(-2R (R\theta_1 - R\theta_2) + 2R (R\theta_2 - R\theta_3)\right)$$

(3) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} = 0 = \ddot{\theta}_3 m R^2 - \tfrac{1}{2} k \left(-2R (R\theta_2 - R\theta_3) + 2R (R\theta_3 - R\theta_1)\right)$$

Now, for these sorts of equations, we usually guess solutions of the sinusoidal type

$$\theta_1 = \theta_{10} e^{i\omega t} \qquad \theta_2 = \theta_{20} e^{i\omega t} \qquad \theta_3 = \theta_{30} e^{i\omega t}$$

where ω is importantly the same for all of them since we are looking for normal modes. Then differentiate, and get:

$$\ddot{\theta}_1 = -\omega^2 \theta_{10} e^{i\omega t} \qquad \ddot{\theta}_2 = -\omega^2 \theta_{20} e^{i\omega t} \qquad \ddot{\theta}_3 = -\omega^2 \theta_{30} e^{i\omega t}$$

Next, write the equations of motion in terms of these general soln's:

$$-\omega^2 \theta_{10} e^{i\omega t} m R^2 = k R^2 \left[(\theta_{10} e^{i\omega t} - \theta_{20} e^{i\omega t}) - (\theta_{30} e^{i\omega t} - \theta_{10} e^{i\omega t}) \right]$$

(4) $$-\omega^2 \theta_{10} m R^2 = k R^2 \left((\theta_{10} - \theta_{20}) - (\theta_{30} - \theta_{10}) \right) = 2kR^2 \theta_{10} - kR^2 \theta_{20} - kR^2 \theta_{30}$$

similarly

(5) $$-\omega^2 \theta_{20} m R^2 = k R^2 \left((\theta_{20} - \theta_{10}) + (\theta_{20} - \theta_{30}) \right) = 2kR^2 \theta_{20} - kR^2 \theta_{10} - kR^2 \theta_{30}$$

(6) $$-\omega^2 \theta_{30} m R^2 = k R^2 \left((\theta_{30} - \theta_{20}) + (\theta_{30} - \theta_{10}) \right) = 2kR^2 \theta_{30} - kR^2 \theta_{20} - kR^2 \theta_{10}$$

Rewrite as

$$-\omega^2 R^2 m \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\hat{T}} \begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix} = k R^2 \underbrace{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_{\hat{U}} \begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix}$$ ✓

or in other words $-\omega^2 m \hat{T} \vec{\theta} = k \hat{U} \vec{\theta} \implies \hat{T}^{-1} \hat{U} \vec{\theta} = -\frac{\omega^2 m}{k} \vec{\theta}$ where $\vec{\theta} = \begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix}$

Set $-\frac{\omega^2 m}{k} \equiv \lambda$ and $\hat{T}^{-1} \hat{U} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \hat{M}$

Solve the eigenvalue problem! $\hat{M} \vec{\theta} = \lambda \vec{\theta}$ (use Mathematica! $\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 0$)

Thus $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{3k}{m}}$, $\omega_3 = 0$ ✓

$$\vec{\theta}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{\theta}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{\theta}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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so the normal modes are the eigenvectors, and their frequencies, the corresponding ω's.

(normalize using $(A\vec{\theta}_1 \hat{T} (A)\vec{\theta}_1 = 1$

normal mode: $\vec{\theta}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{\theta}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{\theta}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

frequency: $\omega_1 = \sqrt{\frac{3k}{m}}$ $\omega_2 = \sqrt{\frac{3k}{m}}$ $\omega_3 = 0$

The zero frequency corresponds to the masses all rotating in unison around the circle w/o oscillating.

part a

I write down the Potential and Kinetic Energy Tensors, and find the Eigenvalues of the secular equation, as shown in the handwritten portion.

```
In[51]:= U = k R^2 {{2, -1, -1}, {-1, 2, -1}, {-1, -1, 2}};
         tT = R^2 m {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
         Eigenvalues[{{2, -1, -1}, {-1, 2, -1}, {-1, -1, 2}}]
         Eigenvectors[{{2, -1, -1}, {-1, 2, -1}, {-1, -1, 2}}]
Out[53]= {3, 3, 0}
Out[54]= {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}
```

```
Solve[Det[U - ω^2 iI] == 0, ω]
```

$$\left\{\{\omega \to 0\}, \{\omega \to 0\}, \left\{\omega \to -\frac{\sqrt{3}\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to -\frac{\sqrt{3}\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{3}\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{3}\sqrt{k}}{\sqrt{m}}\right\}\right\}$$

Normalize

```
In[34]:= Solve[A^2 * {-1, 0, 1}.{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}.{-1, 0, 1} == 1, A]
         Solve[A^2 * {-1, 1, 0}.{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}.{-1, 1, 0} == 1, A]
         Solve[A^2 * {1, 1, 1}.{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}.{1, 1, 1} == 1, A]
```

Out[34]= $\left\{\left\{A \to -\frac{1}{\sqrt{2}}\right\}, \left\{A \to \frac{1}{\sqrt{2}}\right\}\right\}$

Out[35]= $\left\{\left\{A \to -\frac{1}{\sqrt{2}}\right\}, \left\{A \to \frac{1}{\sqrt{2}}\right\}\right\}$

Out[36]= $\left\{\left\{A \to -\frac{1}{\sqrt{3}}\right\}, \left\{A \to \frac{1}{\sqrt{3}}\right\}\right\}$

Solution Continues On The Next Page…

(b) Now $k \to k + \delta k$ for one mass:

$$U = \frac{1}{2}(k+\delta k)(R\theta_3 - R\theta_1)^2 + \frac{1}{2}k(R\theta_1 - R\theta_2)^2 + \frac{1}{2}k(R\theta_2 - R\theta_3)^2$$

As before (see mathematica) we get the following three Eq's of motion:

$$\ddot{\theta}_1 mR^2 + kR^2(\theta_1 - \theta_2) - R^2(k+\delta k)(\theta_3 - \theta_1) = 0$$

$$\ddot{\theta}_2 mR^2 - kR^2(\theta_1 - \theta_2) + kR^2(\theta_2 - \theta_3) = 0$$

$$\ddot{\theta}_3 mR^2 - kR^2(\theta_2 - \theta_3) + R^2(k+\delta k)(\theta_3 - \theta_1) = 0$$

→ Guess the same solutions as before!

$$\theta_1 = \theta_{10} e^{i\omega t} \quad \theta_2 = \theta_{20} e^{i\omega t} \quad \theta_3 = \theta_{30} e^{i\omega t}$$

$$\Rightarrow \ddot{\theta}_1 = -\omega^2 \theta_{10} e^{i\omega t}, \quad \ddot{\theta}_2 = -\omega^2 \theta_{20} e^{i\omega t}, \quad \ddot{\theta}_3 = -\omega^2 \theta_{30} e^{i\omega t}$$

→ write the equations of motion in terms of these...

$$-\omega^2 \theta_{10} mR^2 = -kR^2(\theta_{10} - \theta_{20}) + R^2(k+\delta k)(\theta_{30} - \theta_{10})$$

$$-\omega^2 \theta_{20} mR^2 = kR^2(\theta_{10} - \theta_{20}) - kR^2(\theta_{20} - \theta_{30})$$

$$-\omega^2 \theta_{30} mR^2 = kR^2(\theta_{20} - \theta_{30}) - R^2(k+\delta k)(\theta_{30} - \theta_{10})$$

rewrite as

$$-\omega^2 \cancel{R^2} m \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\hat{T}} \vec{\theta} = \cancel{R^2} \underbrace{\begin{bmatrix} -(2k+\delta k) & k & k+\delta k \\ k & -2k & k \\ k+\delta k & k & -(2k+\delta k) \end{bmatrix}}_{\hat{U}} \vec{\theta} \rightarrow -\omega^2 m \hat{T}\vec{\theta} = \hat{U}\vec{\theta}$$

$$\Rightarrow \hat{T}^{-1}\hat{U}\,\vec{\theta} = -\omega^2 m \vec{\theta}$$

Solving the eigenvalue problem: $\hat{M}\vec{\theta} = \lambda\vec{\theta}$ where $\hat{M} \equiv \hat{T}^{-1}\hat{U}$ and $\lambda = -\omega^2 m$

$$\lambda = 0,\ -3k,\ -3k - 2\delta k \Rightarrow \boxed{\omega_1 = 0,\quad \omega_2 = \sqrt{\frac{3k}{m}},\quad \omega_3 = \sqrt{\frac{3k+\delta k}{m}}}$$

Mathematica

$$\boxed{\vec{\theta}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{\theta}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \vec{\theta}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

Only one normal mode (#2) changes!

part b

This follows the exact same procedure, except for the case where we have a small shift in spring constant k

```
In[97]:= U = {{- (2 k + δk), k, k + δk}, {k, - 2 k, k}, {k + δk, k, - (2 k + δk)}};
tT = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
Eigenvalues[Inverse[tT].U]
Eigenvectors[Inverse[tT].U]

Out[99]= {0, -3 k, -3 k - 2 δk}

Out[100]= {{1, 1, 1}, {1, -2, 1}, {-1, 0, 1}}
```

Normalize

```
In[78]:= Solve[A^2 * {1, 1, 1}.tT.{1, 1, 1} == 1, A]
Solve[A^2 * {1, -2, 1}.tT.{1, -2, 1} == 1, A]
Solve[A^2 * {-1, 0, 1}.tT.{-1, 0, 1} == 1, A]
```

Out[78]= $\left\{\left\{A \to -\frac{1}{\sqrt{3}}\right\}, \left\{A \to \frac{1}{\sqrt{3}}\right\}\right\}$

Out[79]= $\left\{\left\{A \to -\frac{1}{\sqrt{6}}\right\}, \left\{A \to \frac{1}{\sqrt{6}}\right\}\right\}$

Out[80]= $\left\{\left\{A \to -\frac{1}{\sqrt{2}}\right\}, \left\{A \to \frac{1}{\sqrt{2}}\right\}\right\}$

(c) Yeesh... ok, now $m \to \delta m$ for one mass.

$T = \frac{1}{2}(m+\delta m)R^2\dot{\theta}_1^2 + \frac{1}{2}mR^2\dot{\theta}_2^2 + \frac{1}{2}mR^2\dot{\theta}_3^2$, U same as in part a.

Now we find the equations of motion again

$$0 = \ddot{\theta}_1 (m+\delta m)R^2 - kR^2\left((\theta_1-\theta_2) - (\theta_3-\theta_1)\right)$$

$$0 = \ddot{\theta}_2 mR^2 - kR^2\left((\theta_2-\theta_1) + (\theta_2-\theta_3)\right)$$

$$0 = \ddot{\theta}_3 mR^2 - kR^2\left((\theta_3-\theta_2) + (\theta_3-\theta_1)\right)$$

Guess solutions of the form $\theta_1 = \theta_{10}e^{i\omega t}$, $\theta_2 = \theta_{20}e^{i\omega t}$, $\theta_3 = \theta_{30}e^{i\omega t}$

so, $\ddot{\theta}_1 = -\omega^2\theta_{10}e^{i\omega t}$, $\ddot{\theta}_2 = -\omega^2\theta_{20}e^{i\omega t}$, $\ddot{\theta}_3 = -\omega^2\theta_{30}e^{i\omega t}$

Plug these in to get:

$$-\omega^2(m+\delta m)\cancel{R^2}\,\theta_{10} = 2k\cancel{R^2}\theta_{10} - k\cancel{R^2}\theta_{20} - kR^2\theta_{30}$$

$$-\omega^2 m R^2\theta_{20} = 2k\cancel{R^2}\theta_{20} - k\cancel{R^2}\theta_{10} - k\cancel{R^2}\theta_{30}$$

$$-\omega^2\theta_{30} m \cancel{R^2} = 2k\cancel{R^2}\theta_{30} - k\cancel{R^2}\theta_{20} - k\cancel{R^2}\theta_{10}$$

Rewrite in matrix form:

$$-\omega^2\begin{bmatrix} m+\delta m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\end{bmatrix}\begin{bmatrix}\theta_{10}\\ \theta_{20}\\ \theta_{30}\end{bmatrix} = k\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2\end{bmatrix}\begin{bmatrix}\theta_{10}\\ \theta_{20}\\ \theta_{30}\end{bmatrix}$$

Or in other words:

$$-\omega^2\hat{T}\vec{\theta} = k\hat{U}\vec{\theta} \longrightarrow \hat{T}^{-1}\hat{U}\vec{\theta} = \frac{-\omega^2}{k}\vec{\theta}$$

Set $\hat{M} = \hat{T}^{-1}\hat{U}$, and $\lambda = \frac{-\omega^2}{k}$, so we have the eigenvalue

problem: $\hat{M}\vec{\theta} = \lambda\vec{\theta}$

eigenfrequencies

eigenvalues: $\lambda = 0, \frac{3}{m}, \frac{3m+\delta m}{(m+\delta m)m}$ $\Rightarrow$ $\boxed{\omega_1 = 0,\ \omega_2 = \sqrt{\frac{3k}{m}},\ \omega_3 = \sqrt{\frac{k(3m+\delta m)}{m(m+\delta m)}}}$

eigenvectors: $\vec{\theta}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{\theta}_2 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$, $\vec{\theta}_3 = \begin{bmatrix}\frac{-2m}{m+\delta m}\\1\\1\end{bmatrix}$

apply normalization $(\vec{\theta}\hat{T}\vec{\theta} = 1)$

The corresponding normal modes are

$$\boxed{\vec{\theta}_1 = \frac{1}{\sqrt{3m+\delta m}}\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \vec{\theta}_2 = \frac{1}{\sqrt{2m}}\begin{bmatrix}0\\-1\\1\end{bmatrix},\quad \vec{\theta}_3 = \frac{1}{\sqrt{2m + \frac{4m^2}{m+\delta m}}}\begin{bmatrix}\frac{-2m}{m+\delta m}\\1\\1\end{bmatrix}}$$

part c

And... one more time. This time the little shift is in mass

In[105]:=
```
tT = {{m + δm, 0, 0}, {0, m, 0}, {0, 0, m}};
U = {{2, -1, -1}, {-1, 2, -1}, {-1, -1, 2}};
```

In[107]:=
```
Eigenvalues[Inverse[tT].U]
Eigenvectors[Inverse[tT].U]
```

Out[107]= $\left\{0,\ \frac{3}{m},\ \frac{3\,m + \delta m}{m\,(m + \delta m)}\right\}$

Out[108]= $\left\{\{1, 1, 1\},\ \{0, -1, 1\},\ \left\{-\frac{2\,m}{m + \delta m},\ 1,\ 1\right\}\right\}$

Normalize

In[115]:=
$$\text{Solve}\left[A^2 * \{1, 1, 1\}.\text{tT}.\{1, 1, 1\} == 1, A\right]$$
$$\text{Solve}\left[A^2 * \{0, -1, 1\}.\text{tT}.\{0, -1, 1\} == 1, A\right]$$
$$\text{Solve}\left[A^2 * \left\{-\frac{2\,m}{m + \delta m}, 1, 1\right\}.\text{tT}.\left\{-\frac{2\,m}{m + \delta m}, 1, 1\right\} == 1, A\right]$$

Out[115]= $\left\{\left\{A \to -\frac{1}{\sqrt{3\,m + \delta m}}\right\},\ \left\{A \to \frac{1}{\sqrt{3\,m + \delta m}}\right\}\right\}$

Out[116]= $\left\{\left\{A \to -\frac{1}{\sqrt{2}\,\sqrt{m}}\right\},\ \left\{A \to \frac{1}{\sqrt{2}\,\sqrt{m}}\right\}\right\}$

Out[117]= $\left\{\left\{A \to -\frac{1}{\sqrt{2\,m + \frac{4\,m^2}{m + \delta m}}}\right\},\ \left\{A \to \frac{1}{\sqrt{2\,m + \frac{4\,m^2}{m + \delta m}}}\right\}\right\}$