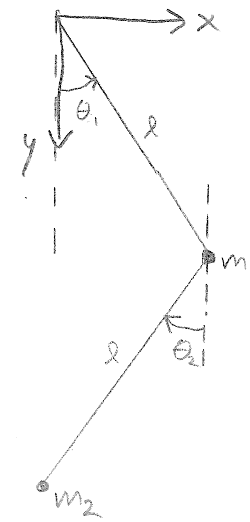
Goldstein 6-4 I begin by finding the Potential and Kinetic Energies (which we already found in problem 1-22) In that problem we had l_1 and l_2. Now $l_1, l_2 \to l$.



Kinetic Energy

$$T = \frac{1}{2} m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l^2 \dot{\theta}_1^2 + l^2 \dot{\theta}_2^2 - 2 l^2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \right)$$

$$T = \frac{1}{2} m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 - 2\cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \right)$$

$$T = \frac{1}{2}(m_1 + m_2) l^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l^2 \dot{\theta}_2^2 - m_2 l^2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2$$

Since we deal w/ small oscillations, $\cos(\theta_1 - \theta_2) \approx 1$

$$T = \frac{1}{2}(m_1 + m_2) l^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l^2 \dot{\theta}_2^2 - m_2 l^2 \dot{\theta}_1 \dot{\theta}_2$$

Thus $\overleftrightarrow{T} = l^2 \begin{bmatrix} (m_1 + m_2) & +m_2 \\ +m_2 & m_2 \end{bmatrix}$ (using $T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j$)

Potential Energy

$$U = -m_1 g l \cos\theta_1 - m_2 g l (\cos\theta_1 + \cos\theta_2)$$

$$U = -(m_1 + m_2) g l \cos\theta_1 - m_2 g l \cos\theta_2$$

w/ small oscillations, we use $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U \approx -(m_1 + m_2) g l + \frac{1}{2}(m_1 + m_2) g l \theta_1^2 - m_2 g l + \frac{1}{2} m_2 g l \theta_2^2$$

$$= -(m_1 + 2m_2) g l + \frac{1}{2}(m_1 + m_2) g l \theta_1^2 + \frac{1}{2} m_2 g l \theta_2^2$$

Thus $\overleftrightarrow{U} = gl \begin{bmatrix} (m_1 + m_2) & 0 \\ 0 & m_2 \end{bmatrix}$ ✓ (using $U = \frac{1}{2} U_{ij} \eta_i \eta_j$)

Secular Equation to find frequency of vibration: (using 6.53)

$$\left| \overleftrightarrow{U} - \omega^2 \overleftrightarrow{T} \right| = \begin{vmatrix} gl(m_1 + m_2) - l(m_1 + m_2)\omega^2 & l^2 \omega_2 m_2 \\ l^2 m_2 \omega^2 & g l m_2 - m_2 \omega^2 l^2 \end{vmatrix} = 0$$

(mathematica!)

Solve this, and get $\omega_1^2 = \frac{g}{l} + \frac{g}{l}\frac{m_2}{m_1} + \frac{g}{l}\frac{\sqrt{(m_1 + m_2) m_2}}{m_1} = \frac{g}{l}\left(1 + \frac{m_2}{m_1} + \frac{\sqrt{(m_1 + m_2) m_2}}{m_1}\right)$

$$\omega_2^2 = \frac{g}{l}\left(1 + \frac{m_2}{m_1} - \frac{\sqrt{(m_1 + m_2) m_2}}{m_1}\right)$$

Normal Modes

Use Mathematica to solve (and simplify) $(\overleftrightarrow{U} - \omega_1^2 \overleftrightarrow{T}) \cdot \vec{\varepsilon} = 0$ and $(\overleftrightarrow{U} - \omega_2^2 \overleftrightarrow{T}) \cdot \vec{\varepsilon} = 0$ let the eigenvector $\vec{\varepsilon} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

For the two frequencies we get $\left(\frac{\alpha}{\beta}\right)_1 = \sqrt{\frac{m_2}{m_1 + m_2}}$ and $\left(\frac{\alpha}{\beta}\right)_2 = -\sqrt{\frac{m_2}{m_1 + m_2}}$

(That is, we can only solve for the ratios)

But that is all we need. The eigenvectors then become. (A_1, A_2 are normalization constants)

$$\vec{\varepsilon}_1 = A_1\begin{bmatrix}\sqrt{m_2/(m_1+m_2)} \\ 1\end{bmatrix} \qquad \vec{\varepsilon}_2 = A_2\begin{bmatrix}-\sqrt{m_2/(m_1+m_2)} \\ 1\end{bmatrix}$$

The normalization condition is $A_1 \tilde{\varepsilon}_1 \hat{T} A_1 \varepsilon_1 = 1$ and $A_2^2 \tilde{\varepsilon}_2 \hat{T} \varepsilon_2 = 1$
(see mathematica).

$$\boxed{\omega_1 = \sqrt{\frac{g}{\ell}\left(1 + \frac{m_2}{m_1} + \frac{\sqrt{(m_1+m_2)m_2}}{m_1}\right)} \qquad \omega_2 = \sqrt{\frac{g}{\ell}\left(1 + \frac{m_2}{m_1} - \frac{\sqrt{(m_1+m_2)m_2}}{m_1}\right)}}$$

$$\vec{\varepsilon}_1 = \underbrace{\frac{1}{\sqrt{\frac{2m_2(m_1+m_2+\sqrt{m_2(m_1+m_2)})}{m_1+m_2}}}}_{\equiv \gamma^+}\begin{bmatrix}1 \\ \sqrt{\frac{m_1+m_2}{m_2}}\end{bmatrix}, \qquad \vec{\varepsilon}_2 = \underbrace{\frac{1}{\sqrt{\frac{2m_2(m_1+m_2-\sqrt{m_2(m_1+m_2)})}{m_1+m_2}}}}_{\equiv \gamma^-}\begin{bmatrix}1 \\ -\sqrt{\frac{m_1+m_2}{m_2}}\end{bmatrix}$$

In the limit that the lower mass is tiny compared to the upper one, $m_2 \ll m_1$, then

$$\omega_1 = \sqrt{\frac{g}{\ell}\left(1 + \underbrace{\frac{m_2}{m_1}}_{\sim 0} + \sqrt{\frac{m_1 m_2}{m_1^2} + \underbrace{\frac{m_2^2}{m_1^2}}_{\sim 0}}\right)}$$

Leaving only the lowest order terms,

$$\omega_1 \approx \sqrt{\frac{g}{\ell}}\left(\sqrt{1+\sqrt{\frac{m_2}{m_1}}}\right) \qquad \text{Similarly} \qquad \omega_2 \approx \sqrt{\frac{g}{\ell}}\left(\sqrt{1-\sqrt{\frac{m_2}{m_1}}}\right)$$

These are very close since $\sqrt{\frac{m_2}{m_1}} \ll 1$, and $\omega_1 \approx \omega_2 \approx \sqrt{\frac{g}{\ell}}$, which is the frequency of the simple top pendulum with no bottom pendulum.

Now, we are given the initial conditions that $\theta_1(0) = \theta_0$, and $\theta_2(0) = \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$

Our equations for the angles of the two masses as functions of time are

$$\vec{\theta}(t) = C_1 \vec{\varepsilon}_1 e^{-i\omega_1 t} + C_2 \vec{\varepsilon}_2 e^{-i\omega_2 t}$$

so

$$\theta_2(t) = C_1 \cos(\omega_1 t) + C_2 \cos(\omega_2 t)$$

$$\theta_1(t) = C_1\sqrt{\frac{m_1+m_2}{m_2}}\cos(\omega_1 t) + C_2\sqrt{\frac{m_1+m_2}{m_2}}\cos(\omega_2 t)$$

$$\theta_2(0) = 0 = C_1 - C_2 \qquad C_1 = C_2 \equiv C$$

$$\theta_1(0) = \theta_0 \Rightarrow \qquad C = \left(\frac{1}{2}\theta_0\sqrt{\frac{m_2}{m_1+m_2}}\right)$$

$$\longrightarrow \quad \theta_1(t) = \frac{\theta_0}{2}\sqrt{\frac{m_2}{m_1+m_2}}(\cos\omega_1 t - \cos\omega_2 t)$$

$$\theta_2(t) = \frac{\theta_0}{2}(\cos\omega_1 t + \cos\omega_2 t)$$

Choosing values for m_1, m_2, and ℓ, I can now plot this in Mathematica (see attached), and the characteristic Shape associated with Beats can easialy be seen. For the $m_2 \gg m_1$ regiem, the behavior is quite noticeable (especially with Small ℓ-lengths).

10 good plot

Define the tensors of Kinetic and Potential Energy

In[158]:= $\mathbf{Ttens} = \{\{m1 + m2, -m2\}, \{-m2, m2\}\};$
$\mathbf{Utens} = \{\{m1 + m2, 0\}, \{0, m2\}\};$

In[160]:= $\mathbf{Assuming}[m1 > 0 \; \&\& \; m2 > 0 \; \&\& \; m1 \in \mathbf{Reals} \; \&\& \; m2 \in \mathbf{Reals},$
$\mathbf{Eigenvalues}[\mathbf{Inverse}[\mathbf{Ttens}].\mathbf{Utens}]]$
$\mathbf{Assuming}[m1 > 0 \; \&\& \; m2 > 0 \; \&\& \; m1 \in \mathbf{Reals} \; \&\& \; m2 \in \mathbf{Reals},$
$\mathbf{Eigenvectors}[\mathbf{Inverse}[\mathbf{Ttens}].\mathbf{Utens}]] \; // \; \mathbf{FullSimplify}$

Out[160]= $$\left\{\frac{m1 + m2 - \sqrt{m1\, m2 + m2^2}}{m1}, \frac{m1 + m2 + \sqrt{m1\, m2 + m2^2}}{m1}\right\}$$

Out[161]= $$\left\{\left\{-\frac{m2}{\sqrt{m2\,(m1 + m2)}}, 1\right\}, \left\{\frac{m2}{\sqrt{m2\,(m1 + m2)}}, 1\right\}\right\}$$

In[163]:= $$\mathbf{Solve}\left[A^2 \left(\left\{\frac{-m2}{\sqrt{m2\,(m1+m2)}}, 1\right\}.\mathbf{Ttens}.\left\{\frac{-m2}{\sqrt{m2\,(m1+m2)}}, 1\right\}\right) == 1, A\right] \; // \; \mathbf{FullSimplify}$$
$$\mathbf{Solve}\left[A^2 \left(\left\{\frac{m2}{\sqrt{m2\,(m1+m2)}}, 1\right\}.\mathbf{Ttens}.\left\{\frac{m2}{\sqrt{m2\,(m1+m2)}}, 1\right\}\right) == 1, A\right] \; // \; \mathbf{FullSimplify}$$

Out[163]= $$\left\{\left\{A \to -\frac{1}{\sqrt{2}\sqrt{\frac{m2\left(m1+m2+\sqrt{m2\,(m1+m2)}\right)}{m1+m2}}}\right\}, \left\{A \to \frac{1}{\sqrt{2}\sqrt{\frac{m2\left(m1+m2+\sqrt{m2\,(m1+m2)}\right)}{m1+m2}}}\right\}\right\}$$

Out[164]= $$\left\{\left\{A \to -\frac{1}{\sqrt{2}\sqrt{\frac{m2\left(m1+m2-\sqrt{m2\,(m1+m2)}\right)}{m1+m2}}}\right\}, \left\{A \to \frac{1}{\sqrt{2}\sqrt{\frac{m2\left(m1+m2-\sqrt{m2\,(m1+m2)}\right)}{m1+m2}}}\right\}\right\}$$

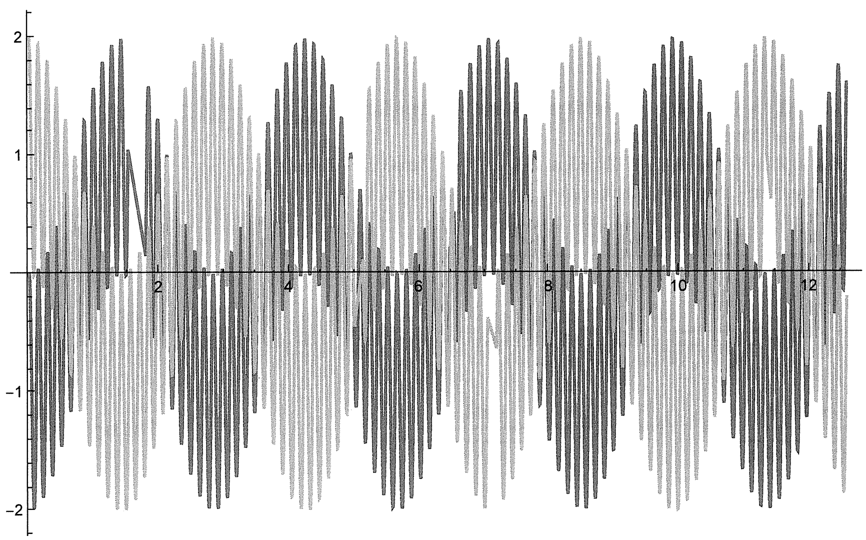
Beats

A qualitative look at the beat frequencies. Note the envelope wave which has a frequency known as the beat frequency.

In[185]:= $$\omega 1 = \sqrt{\frac{g}{\iota}\,\frac{m1 + m2 + \sqrt{m1\, m2 + m2^2}}{m1}}; \qquad \omega 2 = \sqrt{\frac{g}{\iota}\,\frac{m1 + m2 - \sqrt{m1\, m2 + m2^2}}{m1}};$$

In[287]:= $m1 = 1;$
$m2 = 100;$
$g = 9.8;$
$\iota = 1;$

In[300]:= $$\mathbf{Plot}\left[\left\{\sqrt{\frac{m2}{m1 + m2}}\,(\mathbf{Cos}[\omega 1\, t] - \mathbf{Cos}[\omega 2\, t]), (\mathbf{Cos}[\omega 1\, t] + \mathbf{Cos}[\omega 2\, t])\right\},\right.$$
$$\left.\{t, 0, 4\pi\}, \mathbf{PlotRange} \to \mathbf{All}\right]$$

Out[300]=


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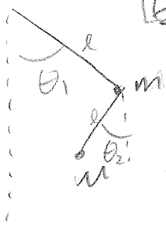
$\boxed{6-4}$ improved from rule sol'n.



$$T = \frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{2} M \left[\ell^2 \dot{\theta}_1^2 + \ell^2 \dot{\theta}_2^2 + 2\ell^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$U = -mg\ell\cos\theta_1 - Mg\ell(\cos\theta_1 + \cos\theta_2)$$

$$\text{so } T = \frac{1}{2}\left[(m\ell^2 + M\ell^2)\dot{\theta}_1^2 + M\ell^2\dot{\theta}_2^2 + 2M\ell^2 \dot{\theta}_1\dot{\theta}_2 \underbrace{\cos(\theta_1 - \theta_2)}_{\approx 1} \right]$$

$$T = \frac{1}{2}\left[\ell^2(m+M)\dot{\theta}_1^2 + \ell^2 M \dot{\theta}_2^2 + M^2\dot{\theta}_1\dot{\theta}_2 + M^2\dot{\theta}_2\dot{\theta}_1 \right]$$

$\theta_1 - \theta_2$ is small, so

$$\hat{T} = \begin{bmatrix} m\ell^2 + M\ell^2 & M\ell^2 \\ M\ell^2 & M\ell^2 \end{bmatrix} = \ell^2 \begin{bmatrix} m+M & M \\ M & M \end{bmatrix}$$

$$U \approx -mg\ell\left(1 - \frac{\theta_1^2}{2}\right) - Mg\ell\left(1 - \frac{\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right)$$

$$U \approx \frac{mg\ell}{2}\theta_1^2 + \frac{Mg\ell}{2}\theta_1^2 + \frac{Mg\ell}{2}\theta_2^2$$

$$\hat{U} = \begin{bmatrix} g\ell(m+M) & 0 \\ 0 & Mg\ell \end{bmatrix} = g\ell \begin{bmatrix} m+M & 0 \\ 0 & M \end{bmatrix}$$

solve $\det(\hat{U} - \omega^2 \hat{T}) = 0$ let $\mu \equiv M + m$,

$$\boxed{\omega_- = \sqrt{\frac{g}{\ell}}\sqrt{\frac{\mu - \sqrt{M\mu}}{m}}, \quad \omega_+ = \sqrt{\frac{g}{\ell}}\sqrt{\frac{\mu + \sqrt{M\mu}}{m}}}$$

$$(\hat{U} - \omega_\pm^2 \hat{T})\vec{a}_\pm = 0 \qquad \vec{a}_+ = \begin{bmatrix} a_1^+ \\ a_2^+ \end{bmatrix}, \quad \vec{a}_- = \begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix}$$

$$\hat{U}\vec{a}_\pm = \omega_\pm^2 \hat{T}\vec{a}_\pm$$

inverse $\rightarrow$ $\hat{T}^{-1}\hat{U}\,\vec{a}_\pm = \omega_\pm^2 a_\pm \overset{\text{Mathematica}}{\Longrightarrow} \vec{a}_- = \begin{bmatrix} \frac{M}{\sqrt{M\mu}} \\ 1 \end{bmatrix}, \quad \vec{a}_+ = \begin{bmatrix} \frac{-M}{\sqrt{M\mu}} \\ 1 \end{bmatrix}$

Normalize: $A_\pm \vec{a}_\pm \hat{T} \vec{a}_\pm = 1$ (use MMA to solve for $A_\pm$, the norm constant).

$$A_\pm = \frac{\mu}{2M(\mu \mp \sqrt{M\mu})} \Longrightarrow \boxed{\vec{a}_\pm = \frac{\mu}{2M(\mu \mp \sqrt{M\mu})}\begin{bmatrix} \frac{\mp M}{\sqrt{M\mu}} \\ 1 \end{bmatrix}}$$

$$\frac{1}{\text{kg m}^2}$$