Goldstein 2-24

Given $L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$ and $x = \sum_{j=0} a_j \cos(j\omega t)$

The first step is to calculate the action between $t_1 = 0$, $t_2 = \frac{2\pi}{\omega}$

$$I = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right) dt = \frac{m}{2}\int_0^{2\pi/\omega} \dot{x}^2\,dt - \frac{k}{2}\int_0^{2\pi/\omega} x^2\,dt$$

Note: $\dot{x}^2 = \left(\sum_{j=0} a_j^2 j^2 \omega^2 \sin^2(j\omega t)\right) + \sum_{j\neq i} a_j a_i\, j i\, \omega^2 \sin(j\omega t)\sin(i\omega t)$

$x^2 = \sum_{j=0} a_j^2 \cos^2(j\omega t) + \sum_{j\neq i} a_j a_i \cos(j\omega t)\cos(i\omega t)$

$$I = \frac{m}{2}\int_0^{2\pi/\omega} dt \left[\left(\sum_{j=0} a_j^2 j^2 \omega^2 \sin^2(j\omega t)\right) + \sum_{j\neq i}\left(a_j a_i\, j i\, \omega^2 \sin(j\omega t)\sin(i\omega t)\right)\right]$$

$$-\frac{k}{2}\int_0^{2\pi/\omega} dt \left[\sum_{j=0} a_j^2 \cos^2(j\omega t) + \sum_{j\neq i} a_j a_i \cos(j\omega t)\cos(i\omega t)\right]$$

$$I = \frac{m}{2} a_j^2 j^2 \omega^2 \int_0^{2\pi/\omega} \sum_{j=0} \sin^2(j\omega t)\,dt + \frac{m}{2} a_j a_i\, j i\, \omega^2 \int_0^{2\pi/\omega} \sum_{i\neq j} \sin(j\omega t)\sin(i\omega t)\,dt$$

$$-\frac{k}{2} a_j^2 \int_0^{2\pi/\omega} \sum_{j=0} \cos^2(j\omega t)\,dt - \frac{k}{2} a_j a_i \int_0 \sum_{j\neq i} \cos(j\omega t)\cos(i\omega t)\,dt$$

$$I = \frac{m a_j^2 j^2 \omega^2}{2}\left[\frac{j2\pi}{2\omega}\right] + \frac{m a_j a_i\, j i\, \omega^2}{2}\left[0\right] - \frac{k a_j^2}{2}\left[\frac{j2\pi}{2\omega}\right] - \frac{k a_j a_i}{2}\left[0\right]$$

$$I = \frac{m a_j^2 j^3 \omega^2 \pi}{2\omega} - \frac{k a_j^2 \pi j}{2\omega} = a_j^2\left(\frac{m j^3 \omega \pi}{2} - \frac{k\pi j}{2\omega}\right)$$

The actual solutions of x happen when I is stationary w/ respect to the a_j parameters. Thus we take the a_j derivative of I to get a list of constraints on the a_j's.

$$\frac{\partial I}{\partial a_j} = 0 = a_j\left(m j^3 \omega \pi - k\pi j/\omega\right)$$

We can safely eliminate the possibility that $j = 0$ alone as the solution is assumed to be periodic — not constant.

$$0 = \frac{\pi m}{\omega}\left((j\omega)^2 - \frac{k}{m}\right) a_j$$

Thus...

<u>Either</u> $a_j = 0$, or $a_j \neq 0$ and $(j\omega)^2 - \frac{k}{m} = 0$

But since the j's are (+) integers, if $j \neq i$, then $(i\omega)^2 - \frac{k}{m} \neq 0$. It's simply a different integer. Thus, if $a_j \neq 0$, then $a_i = 0$ if $j \neq i$. There can only be one a_j that isn't equal to zero.

$(j\omega)^2 = \left(\frac{k}{m}\right)$ Then $\omega^2 = \frac{k}{m} \Rightarrow j = 1. \Rightarrow a_1 \neq 0$

Finally: $\boxed{x(t) = a_1 \cos(\omega t)}$

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