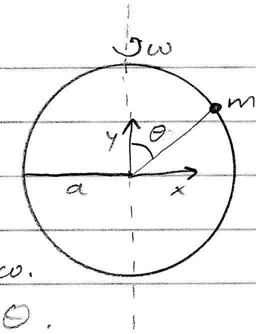
④

Goldstein 2-18

We have a point mass (e.g. a bead) confined on the hoop shown @ right. The mass can slide around (θ can change). Also, the hoop is spinning about the dashed axis w/ angular velocity ω. Happily we only need one generalized coordinate: θ.



Coordinates of the mass

$x = a\sin\theta$, $\dot{x} = \dot{\theta} a\cos\theta$, $\dot{x}^2 = a^2\dot{\theta}^2\cos^2\theta$

$y = a\cos\theta$, $\dot{y} = -\dot{\theta} a\sin\theta$, $\dot{y}^2 = a^2\dot{\theta}^2\sin^2\theta$

Find Lagrangian

T has two componets. One due to θ changing (a.k.a sliding), and the other due to the rotation of the entire hoop).

$T = T_{sliding} + T_{rotating} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}I\omega^2$. $\quad \dot{x}^2+\dot{y}^2 = a^2\dot{\theta}^2$

$I = mx^2 = ma^2\sin^2\theta$

$\Rightarrow T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta$ ✓

$U = mgy = mga\cos\theta$ ✓

$L = T - U = \frac{1}{2}ma^2[\dot{\theta}^2 + \omega^2\sin^2\theta] - mga\cos\theta$ ✓

explain why you are using this → Energy Function (from eq. 2.53)

$h = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L = \dot{\theta}(ma^2\dot{\theta}) - \frac{1}{2}ma^2[\dot{\theta}^2 + a^2\sin^2\theta] + mga\cos\theta$

$= \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta$

Now, to find stationary points, we need to take just the potential terms of the energy function (i.e. not the first one which has $\dot{\theta}$ dependance) — call it h_u.

$h_u = \frac{g}{a}\cos\theta - \frac{1}{2}\omega^2\sin^2\theta$

Solve for stationary points — when h_u doesn't change w/ θ.

$\frac{dh_u}{d\theta} = 0 = -\frac{g}{a}\sin\theta - \omega^2 \cdot 2\cos\theta\sin\theta\left(\frac{1}{2}\right)$

$0 = -\frac{g}{a}\sin\theta - \omega^2\cos\theta\sin\theta$

good

Clearly $\theta = 0$ and $\theta = \pi$ are stationary points.

Otherwise...

$0 = -\frac{g}{a} - \omega^2\cos\theta \Rightarrow \omega^2\cos\theta = -\frac{g}{a} \Rightarrow \cos^{-1}\left(\frac{-g}{\omega^2 a}\right) = \theta$

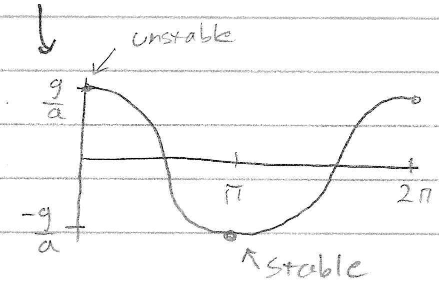
Lets define $\boxed{\omega_0 = \sqrt{\frac{g}{a}}}$ so that $\theta = \cos^{-1}\left(\frac{-\omega_0^2}{\omega^2}\right)$

See next page for case-by-case analysis.

Ben Levy (5)

For stability, determine ~~V''~~ $\frac{d^2 h_n}{d\theta^2}$

units? what is this plot?



$\boxed{\text{Low } \omega: \ \omega < \omega_0}$

→ $\theta = 0$ is stationary, but unstable.

→ $\theta = \pi$ is stationary and stable.

This can be seen on the plot @ right where $\theta = 0$ is a peak, $\theta = \pi$ is a trough.

(meanwhile the "3rd stationary pt" $\theta = \cos^{-1}\left(\frac{-\omega_0^2}{\omega^2}\right)$ is undefined so unimportant.

$\boxed{\text{Transitional } \omega: \ \omega = \omega_0}$

→ now $\theta = \cos^{-1}\left(\frac{-\omega_0^2}{\omega^2}\right) = \cos^{-1}(-1) = \pi$ still stationary & stable

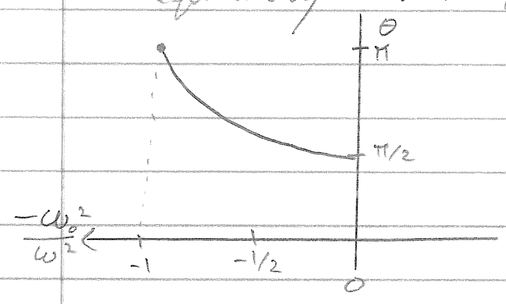
→ $\theta = 0$ is still stationary & unstable.

(Note that we have "double-confirmation" that $\theta = \pi$ works here!"

$\boxed{\text{High } \omega: \ \omega > \omega_0}$

When $\omega > \omega_0$, suddenly the stable pt. starts creeping upward. To see this let's plot $\theta = \cos^{-1}\left(\frac{-\omega_0^2}{\omega_0}\right)$ for $\omega_0 < \omega < \infty$.

Equivalently $-1 < \frac{-\omega_0^2}{\omega^2} < 0$



→ $\theta = \cos^{-1}\left(\frac{-\omega_0^2}{\omega^2}\right)$ is stationary and stable

→ $\theta = \pi$ and $\theta = 0$ are ~~both~~ unstable!

conserved quantities?

8