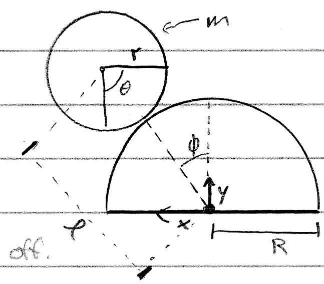
Goldstein 2-14



We are given a hoop of mass m which rolls down the half circle as shown w/o slipping. We are asked to find the point @ which the little hoop falls off.

We use generalized coordinates θ, ϕ, ℓ as shown.

Coordinates of the Hoop

They are identical to those of M in problem #1 where $r \to \ell, \theta \to \phi$.

Find Lagrangian

$$T = \frac{1}{2}m(\dot{\ell}^2 + \ell^2\dot{\phi}^2) + \underbrace{\frac{1}{2}mr^2\dot{\theta}^2}_{\frac{1}{2}I\omega^2}$$ ← same as before w/ extra rotational term

$U = mg\ell\cos\phi$ ← as before

$$L = T - U = \frac{1}{2}m(\dot{\ell}^2 + \ell^2\dot{\phi}^2 + r^2\dot{\theta}^2) - mg\ell\cos\phi$$

Constraints

→ The hoop stays touching the big hoop's surface : $\ell = r + R$

$$g_1(\ell) = r + R - \ell = 0$$

→ The hoop rolls w/o slipping : $(R+r)\phi = r\theta$

$$g_2(\theta,\phi) = r(\theta - \phi) - R\phi$$

Euler-Lagrange Equations

① $$\frac{d}{dt}\left(\frac{\partial L}{\partial \ell}\right) - \frac{\partial L}{\partial \ell} + \lambda_1 \frac{\partial}{\partial \ell} g_1(\ell) + \cancel{\lambda_2 \frac{\partial}{\partial \ell} g_2(\theta,\phi)}^{\,0} = 0$$

$$\Rightarrow m\ddot{\ell} - m\ell\dot{\phi}^2 + mg\cos\phi - \lambda_1 = 0$$

② $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \cancel{\lambda_1 \frac{\partial}{\partial \theta} g_1(\ell)}^{\,0} + \lambda_2 \frac{\partial}{\partial \theta} g_2(\theta,\phi) = 0$$

$$\Rightarrow mr^2\ddot{\theta} + \lambda_2(r+R) = 0$$

③ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} + \cancel{\lambda_1 \frac{\partial}{\partial \phi} g_1(\ell)}^{\,0} + \lambda_2 \frac{\partial}{\partial \phi} g_2(\theta,\phi) = 0$$

$$m\ell^2\ddot{\phi} + 2m\ell\dot{\ell}\phi - mg\ell\sin\phi - \lambda_2(r+R) = 0$$

Initial conditions at $t=0$, $\dot{\ell} = \ddot{\ell} = 0$, $\ell = r + R$, $\theta = \frac{R+r}{r}\phi$, $\ddot{\theta} = \frac{R+r}{r}\ddot{\phi}$

Plugging these into the above: ① $-m\ell\dot{\phi}^2 + mg\cos\phi = \lambda_1$

② $-mr\ddot{\theta} = \lambda_2$ ③ $m(r+R)\ddot{\phi} - mg\sin\phi = \lambda_2$

Ben Levy ③

we should also change ② into a function of ϕ because we are looking for ϕ — not θ.

$$-mr\left(\ddot{\phi}\left(\frac{R+r}{r}\right)\right) = \lambda_2 \Rightarrow \ddot{\phi} = \frac{-\lambda_2}{m(R+r)}$$

So, we have

① $-ml\dot{\phi}^2 + mg\cos\phi = \lambda_1$

② $\ddot{\phi} = -\lambda_2/(m(R+r))$

③ $\ddot{\phi} = \dfrac{\lambda_2 + mg\sin\phi}{m(R+r)}$

Solve the ODEs.

Equate $\ddot{\phi}$ to find λ_2: $\dfrac{-\lambda_2}{m(R+r)} = \dfrac{\lambda_2 + mg\sin\phi}{m(R+r)} \Rightarrow \lambda_2 = \dfrac{-mg\sin\phi}{2}$

no penalty, but note same mistake in earlier problem

Plug into ③: $\ddot{\phi} = \dfrac{(-mg\sin\phi)/2 + mg\sin\phi}{m(r+R)} = \dfrac{mg\sin\phi}{2m(R+r)} = \dfrac{g\sin\phi}{2(R+r)}$

integrate: $\int \ddot{\phi}\, d\phi = \frac{g}{2(R+r)} \int \sin\phi\, d\phi$

(See Goldstein 2.13)

$\frac{1}{2}\dot{\phi}^2 = \frac{1}{2}\left(\frac{g\cos\phi}{R+r} + K\right)$ where K is a constant of integration.

Find K using initial conditions ($\phi(t=0) = \dot{\phi}(t=0) = 0$)

$$0 = \frac{g}{R+r} + K \Rightarrow K = \frac{-g}{R+r}$$

Thus

$$\dot{\phi}^2 = \frac{g\cos\phi - g}{R+r} = \frac{g}{R+r}(\cos\phi - 1)$$

Finally, plug $\dot{\phi}^2$ into equation ①

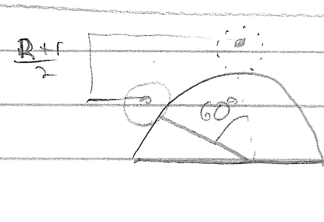
$$\frac{ml\,g(\cos\phi - 1)}{R+r} + mg\cos\phi = \lambda_1 = \frac{m\cancel{(R+r)}\,g(\cos\phi-1)}{\cancel{(R+r)}} + mg\cos\phi$$

$$\lambda_1 = (\cos\phi - 1) + \cos\phi \Rightarrow \lambda_1 = 2\cos\phi - 1$$

Set $\lambda_1 = 0$ to find the point @ which the normal force is zero, indicating the hoop falls off. Solve for ϕ_0, which is the angle when that happens.

$$2\cos\phi_0 - 1 = 0 \Rightarrow \cos(\phi_0) = \frac{1}{2} \Rightarrow \phi_0 = 60^\circ$$

$$\boxed{\text{The Hoop falls off when } \phi = 60^\circ\text{, or when the hoop is } \tfrac{R+r}{2} \text{ below it's starting center height.}}$$



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