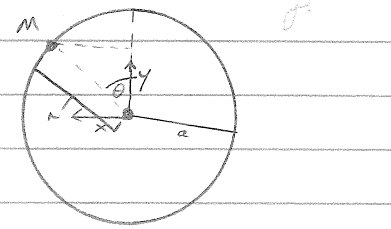
Goldstein 2-13

We begin w/ the diagram at the right in which a particle of mass M falls off of the hoop (Starting @ $\theta = 0$). We are to find the height @ which the particle falls off. We use generalized coordinates r and θ as shown →



Coordinates of Mass $x = r\sin\theta \Rightarrow \dot{x} = \dot{\theta} r\cos\theta + \dot{r}\sin\theta$,

$y = r\cos\theta$, $\dot{y} = -\dot{\theta} r\sin\theta + \dot{r}\cos\theta$,

$\dot{x}^2 = \dot{r}^2\sin^2\theta + \dot{\theta}^2 r^2\cos^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$\dot{y}^2 = \dot{r}^2\cos^2\theta + \dot{\theta}^2 r^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

Find Lagrangian

$T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)$, $U = Mgy = Mgr\cos\theta$

$\Rightarrow L = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mgr\cos\theta$

Constraint: the Mass stays on hoop's surface.

$a - r = 0$. Thus we call $g(r) = a - r = 0$. There is no constraint on θ.

Euler-Lagrange Equations (use 2.22)

① $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \cancel{\lambda_\theta \frac{\partial g}{\partial \theta}}^{\,0} = 0 \Rightarrow \frac{d}{dt}\left[Mr^2\dot{\theta}\right] - Mgr\sin\theta$

$\Rightarrow M\ddot{\theta}r^2 + 2\dot{r}rM\dot{\theta} - Mgr\sin\theta = 0$

② $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} + \lambda_r \frac{\partial g}{\partial r} = 0 \Rightarrow \frac{d}{dt}\left[M\dot{r}\right] - Mr\dot{\theta}^2 + Mg\cos\theta - \lambda_r = 0$

$\Rightarrow M\ddot{r} - Mr\dot{\theta}^2 + Mg\cos\theta - \lambda = 0$

Plug in initial constraints

@ the start $r = a$, and $\dot{r} = \ddot{r} = 0$. This simplifies the E-L equations.

① $M\ddot{\theta}a^2 - Mga\sin\theta = 0 \Rightarrow \ddot{\theta} - \frac{g}{a}\sin\theta = 0$

② $Ma\dot{\theta}^2 - Mg\cos\theta - \lambda_r = 0$

Solve ODEs

careful, $\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\dot{\theta} \Rightarrow \int \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int \sin\theta\, d\theta$

X Integrate both sides of ① w/ respect to θ. $\int \ddot{\theta}\, d\theta = \frac{g}{a}\int \sin\theta\, d\theta$

$\Rightarrow \frac{1}{2}\dot{\theta}^2 = -\frac{g}{a}\cos\theta + K$, where K is a constant of integration.

Plug in initial conditions ($\theta(t=0) = 0$, $\dot{\theta}(t=0) = 0$), then $K = g/a$.

Thus ① becomes $\dot{\theta}^2 = -\frac{2g}{a}\cos\theta + 2g/a$

Now plug this in for the $\dot{\theta}$ in equation ②:

$Ma\left(-\frac{2g}{a}\cos\theta + \frac{2g}{a}\right) - Mg\cos\theta = \lambda_r$

The constraint Force $F = \lambda_r \frac{\partial g(r)}{\partial r} = 2gM(\cos\theta - 1) + Mg\cos\theta$

$F = \boxed{3Mg\cos\theta - 2Mg}$

Now, to find the point at which the Mass falls off, set $F = 0$.

8 $0 = 3Mg\cos\theta - 2Mg \Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \boxed{\theta = 48.2}$ ✓

The mass falls off at $\theta = 48.2$, or $y = a\cos(48.2) = \frac{2}{3}a$

or $\boxed{\frac{1}{3}a \text{ below the top of the hoop.}}$