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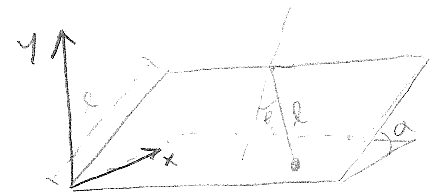
Goldstein 12.11



Get Lagrangian / Hamiltonian

$y = (l - l\cos\theta)\sin\alpha$, $x = (l - l\sin\theta)\cos\alpha$

$U = mgy = mgl(1-\cos\theta)\sin\alpha$

when θ is small, (which it is) $\cos\theta \approx 1 - \frac{\theta^2}{2}$ $\Rightarrow U \approx \frac{mgl}{2}\theta^2\sin\alpha$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \approx \frac{1}{2}ml^2\dot{\theta}^2$

$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{mgl}{2}\theta^2\sin\alpha$

Euler Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 = ml^2\ddot{\theta} + mgl\theta\sin\alpha \Rightarrow \ddot{\theta} + \left(\frac{g}{l}\sin\alpha\right)\theta = 0$$

1 $\Rightarrow \theta = A\cos\left(\sqrt{\frac{g}{l}}\sqrt{\sin\alpha}\, t\right) + B\sin\left(\sqrt{\frac{g}{l}}\sqrt{\sin\alpha}\, t\right)$

Also, $P = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$, so $H = ml^2\dot{\theta}^2 - \frac{1}{2}ml^2\dot{\theta}^2 + \frac{mgl}{2}\theta^2\sin\alpha$

$H = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{mgl}{2}\theta^2\sin\alpha = \frac{P^2}{2ml^2} + \frac{mgl}{2}\theta^2\sin\alpha = E$ 2

Transform to Action/Angle Variables.

$J = \oint p\,dq$ From the hamiltonian, $P^2 = 2ml^2E - m^2l^3g\theta^2\sin\alpha$

so $J = \oint\sqrt{2ml^2E - m^2l^3g\theta^2\sin\alpha}\, d\theta$

$J = \sqrt{2ml^2E}\oint\sqrt{1 - \frac{mlg\sin\alpha}{2E}\theta^2}\, d\theta$

use $\theta = \sqrt{\frac{2E}{mgl\sin\alpha}}\sin\phi \Rightarrow d\theta = \sqrt{\frac{2E}{mgl\sin\alpha}}\cos\phi\, d\phi$

$\therefore J = \sqrt{2ml^2E}\sqrt{\frac{2E}{mgl\sin\alpha}}\int_0^{2\pi}\cos^2\phi\, d\phi = 2\pi E\sqrt{\frac{l}{g\sin\alpha}}$

Rewrite Hamiltonian in terms of J

$E = \frac{J}{2\pi}\sqrt{\frac{g\sin\alpha}{l}} \Rightarrow H = \frac{J}{2\pi}\sqrt{\frac{g\sin\alpha}{l}}$

Now, notice from ①, $A = \theta_{max}$. (A is the amplitude)

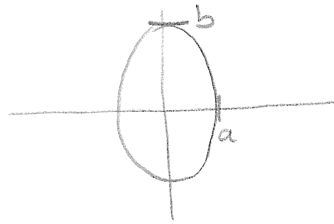
So, what is θ_{max}? It is easy to see by plotting ② in phase space:

$$1 = \frac{p^2}{2ml^2E} + \frac{\theta^2}{\frac{2E}{mgl\sin\alpha}}$$

(rearranging to look like an ellipse where we have $1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$)

Thus the semi-major and semiminor axes (a and b) are given by

$$a = \sqrt{2ml^2E}\ , \quad b = \sqrt{\frac{2E}{mgl\sin\alpha}}$$



Then it's easy to see that $b = \theta_{max} = \sqrt{\frac{2E}{mgl\sin\alpha}}$

Thus $A = \sqrt{\frac{2E}{mgl\sin\alpha}} = \sqrt{\frac{\cancel{2}}{mgl\sin\alpha}}\sqrt{\frac{J}{\cancel{2}\pi}}\sqrt{\frac{g\sin\alpha}{l}}$

$$A = \sqrt{\frac{J}{m l^{3/2}\sqrt{g\sin\alpha}\,\pi}} = \boxed{\sqrt{\frac{J}{m l^{3/2}\pi g^{1/2}}}\,(\sin\alpha)^{-1/4}}$$

Noting finally that this motion is adiabatic, so $J = \text{constant}$, the amplitude varies as a function of $\boxed{(\sin\alpha)^{-1/4}}$

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