#5 Goldstein 12-6 ⓐ Show $H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) - \frac{1}{8}\frac{p^4}{m^3c^2}$

for a relativistic 1-D Harmonic Oscillator.

$$E^2 = (pc)^2 + (mc^2)^2 \Rightarrow E = \sqrt{(pc)^2 + (mc^2)^2} = mc^2\sqrt{1 + \left(\frac{pc}{mc^2}\right)^2}$$

Expand using Binomial theorem:

$$E = mc^2\left(1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + - + - \cdots\right)$$

$$E = mc^2 + \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2}$$ ← Truncate.

$$T = E_{rel} - mc^2 = \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2}$$

$$U = \frac{m\omega^2 q^2}{2}$$ ← Harmonic Oscillator Potential

$$\boxed{H = T + U = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) - \frac{1}{8}\frac{p^4}{m^3c^2}}$$ ✓

ⓑ Take $H_0 = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$, and $\Delta H_1 = -\frac{1}{8}\frac{p^4}{m^3c^2}$

we already know the solution to H_0. It's soluble,

See attached "Appendix" for Derivation → By 10.97: $p = \sqrt{\frac{mJ\omega}{\pi}}\cos(2\pi w)$ using action angle variables (J, w).

Then $\Delta H_1 = -\frac{1}{8}\left(\frac{mJ\omega}{\pi}\right)^2 \cos^4(2\pi w)\frac{1}{m^3c^2} = -\frac{J^2\omega^2\cos^4(2\pi w)}{8\pi^2 mc^2}$

We use $\nu_1 = \overline{\frac{\partial \Delta H_1}{\partial J}} = \frac{-\overset{2\text{ (from derivative)}}{\cancel{J}}\omega^2}{\underset{\pi^2}{8\cancel{\pi}mc^2}}\int_0 \underbrace{\cos^4(2\pi w)\,dw}_{3/8}$

$$\nu_1 = \frac{-3J\omega^2}{64\pi mc^2}, \quad \text{Now note } J = \frac{2\pi E}{\omega}, \text{ so } \nu_1 = \frac{-3E\omega}{32mc^2}$$

Since $\nu_0 = \frac{\omega}{2\pi}$, $$\boxed{\frac{\nu_1}{\nu_0} = \frac{-3E\pi}{16mc^2}}$$ ✓ $-\frac{3}{8}\frac{E}{mc^2}$

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Appendix: Solve the Simple harmonic Oscillator Problem in Action-Angle variables. (important for #4 & #5)

(1) $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} = E$

Solve for $p(q, \text{constants})$.

(2) $p^2 + m^2\omega^2 q^2 = 2mE \Rightarrow p = \sqrt{2mE - m^2\omega^2 q^2}$

(3) $J = \oint P\,dq = \oint \sqrt{2mE - m^2\omega^2 q^2}\, dq$

That's a difficult integral, so lets use the substitution $q = \sqrt{\frac{2E}{m\omega^2}} \sin\theta$ (4)

The integrand becomes: $\sqrt{2mE - 2Em\sin^2\theta} = \sqrt{2mE}\sqrt{1-\sin^2\theta} = \sqrt{2mE}\cos\theta$

dq becomes: $\frac{dq}{d\theta} = \sqrt{\frac{2E}{m\omega^2}}\cos\theta \Rightarrow dq = \sqrt{\frac{2E}{m\omega^2}}\cos\theta\, d\theta$

(5) $J = \int_0^{2\pi} \sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}\cos^2\theta\, d\theta = \frac{2E}{\omega}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{2E\pi}{\omega}$

(6) $\Rightarrow E = \frac{J\omega}{2\pi}$

Thus:

(7) $q = \sqrt{\frac{J}{2\pi m}}\sin\theta$ and plugging (4) into (2), $p = \sqrt{2mE}\cos\theta \Rightarrow p = \sqrt{\frac{mJ\omega}{\pi}}\cos\theta$

Now we know $w = \nu t + \beta$, and the usual solution to the oscillator problem is $q = \sqrt{\frac{J}{2\pi m}}\sin(\omega t + \beta)$, $p = \sqrt{\frac{mJ\omega}{\pi}}\cos(\omega t + \beta)$

Thus $\theta = \omega t + \beta$, and since $\omega t + \beta = 2\pi\nu t + \beta$

Define $\beta \equiv \frac{\beta}{2\pi}$, then $\theta = \omega t + \beta = 2\pi(\nu t + \beta) = 2\pi w$

So

$$\boxed{q = \sqrt{\frac{J}{2\pi m}}\sin(2\pi w) \qquad p = \sqrt{\frac{mJ\omega}{\pi}}\cos(2\pi w)}$$