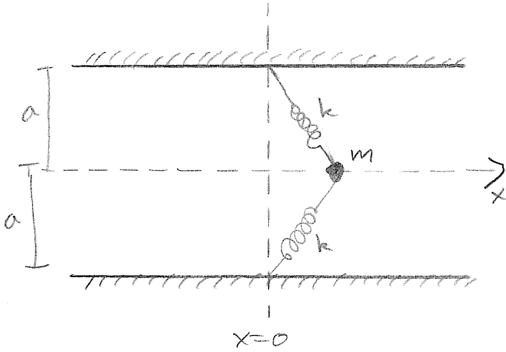
#4 Goldstein 12-5

We are going to analyze this system using Perturbation Theory in order to find $\Delta\nu_1$ : the 1st order correction to the eigen frequency.




(m is confined to move in x-direction)

Begin by writing the Hamiltonian

$T = \frac{1}{2}m\dot{x}^2$ , $U = 2 \times \frac{1}{2}m\left(\sqrt{a^2+x^2} - b\right)^2$

(The potential energy is multiplied by two for two springs)

($\sqrt{a^2+x^2}$ is the length of the spring @ any pt. x, and b is the rest length.)

$L = T - U = \frac{1}{2}m\dot{x}^2 - k\left(\sqrt{a^2+x^2} - b\right)^2$ , and $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

so $H = \dot{x}p - L$

$H = \frac{1}{2}m\dot{x}^2 + k\left(a\sqrt{1+\left(\frac{x}{a}\right)^2} - b\right)^2$

Expand the Hamiltonian, and pick out the Solubal, and first order perturbation terms

use the binomial expansion: $\sqrt{1+y^2} = 1 + \frac{y^2}{2} - \frac{y^4}{8} + \mathcal{O}(y^6)$

Thus, $a\sqrt{1+\left(\frac{x}{a}\right)^2} \approx a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$

Then $H = \frac{1}{2}m\dot{x}^2 + k\left(a - b + \frac{x^2}{2a} - \frac{x^4}{8a^3}\right)^2$

Expand: $H = \frac{1}{2}m\dot{x}^2 + \underbrace{ka^2 - 2kab + kb^2}_{\text{constants.}} + kx^2 - \frac{bkx^2}{a} + \frac{bkx^4}{4a^3} + \mathcal{O}(x^6)$

Let's drop the constants b/c we are only going to differentiate the Hamiltonian. Also let's drop higher order terms in x b/c we only want first order correction.

$H = \frac{1}{2}m\dot{x}^2 + kx^2 - \frac{bkx^2}{a} + \frac{bkx^4}{4a^3} = \frac{1}{2}m\dot{x}^2 + kx^2\left(1 - \frac{b}{a}\right) + \frac{bkx^4}{4a^3}$

$H = \frac{P^2}{2m} + \frac{1}{2}\left[2k\left(1 - \frac{b}{a}\right)\right]x^2 + \frac{kb}{4a^3}x^4$

Now set $\mathcal{K} \equiv 2k\left(1 - \frac{b}{a}\right)$ — the modified spring constant, and $\omega_0^2 = \frac{\mathcal{K}}{m}$

$H = \frac{P^2}{2m} + \frac{1}{2}m\omega_0^2x^2 + \frac{kb}{4a^3}x^4$

The first two terms of this hamiltonian are the unperturbed Simple Harmonic Oscillator Hamiltonian, and the 3rd term is the first order correction:

$H = H_0 + \Delta H_1$

$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega_0^2x^2$ $\quad \Delta H_1 = \frac{kb}{4a^3}x^4$ ✓

The simple Harmonic Oscillator described by Hamiltonian $H_0$ is soluble. The well-known solution in action-angle variables J and $w$ (not to be confused with $\omega_0$) is given, for example, by 10.96:

$$x_0 = \sqrt{\frac{J}{m\pi\omega_0}} \sin(2\pi w)$$

← See derivation in attached "Appendix"

The game that we play in Perturbation Theory is to plug our zeroth-order trajectory into $\Delta H_1$, and then we could get $x_1$. Of corse, we don't need to go that far. We just need eigenfrequencies.

$$\Delta H_1 = \frac{kb}{4a^3} x_0^4 = \frac{kb}{4a^3} \frac{J^2}{m^2\pi^2\omega_0^2} \sin^4(2\pi w)$$

$$\nu_1 = \overline{\frac{\partial \Delta H_1}{\partial J}} = \frac{kb}{2a^3} \frac{J}{m^2\pi^2\omega_0^2} \overline{\sin^4(2\pi w)}$$

setting $u = 2\pi w$, $dw = \frac{1}{2\pi} du$ $\Rightarrow$ $\overline{\sin^4(2\pi w)} = \frac{1}{1-0}\int_0^{2\pi} \sin^4(u)\frac{1}{2\pi} du = \frac{3}{8}$

Thus

$$\nu_1 = \frac{kbJ}{16a^3m^2\pi^2\omega_0^2}$$

Noting that, from 10.96, the max amplitude is $A \equiv \sqrt{\frac{J}{m\pi\omega_0}}$, we get

$$J = A^2 m\pi\omega_0$$

$$\nu_1 = \frac{3kbA^2\cancel{m\pi\omega_0}}{16a^3m^{\cancel{2}}\pi^{\cancel{2}}\omega_0^{\cancel{2}}} = \frac{3}{16}\frac{kbA^2}{a^3m\pi^2\omega_0}$$

and using $k = \frac{K}{2(1-\frac{b}{a})}$, we get $\boxed{\nu_1 = \frac{3}{16}\frac{KbA^2}{m^2\pi\omega_0 a^2(a-b)}}$

This is our first order correction to the unpreturbed frequency $\nu_0 = \sqrt{\frac{k}{m}}$. Note that as a and b get close, $\nu_1$'s denomiter gets small, so $\nu_1$ dominates compared to $\nu_0$. This makes sense since when the springs are no longer stretched, we lose the harmonic oscillator behavior.

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Appendix: Solve the Simple harmonic Oscillator Problem in Action-Angle variables. (important for #4 & #5)

(1) $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} = E$

Solve for $p(q, \text{constants})$.

(2) $p^2 + m^2\omega^2 q^2 = 2mE \Rightarrow p = \sqrt{2mE - m^2\omega^2 q^2}$

(3) $J = \oint P\,dq = \oint \sqrt{2mE - m^2\omega^2 q^2}\,dq$

That's a difficult integral, so lets use the substitution $q = \sqrt{\frac{2E}{m\omega^2}}\sin\theta$ (4)

The integrand becomes: $\sqrt{2mE - 2Em\sin^2\theta} = \sqrt{2mE}\sqrt{1-\sin^2\theta} = \sqrt{2mE}\cos\theta$

dq becomes: $\frac{dq}{d\theta} = \sqrt{\frac{2E}{m\omega^2}}\cos\theta \Rightarrow dq = \sqrt{\frac{2E}{m\omega^2}}\cos\theta\,d\theta$

(5) $J = \int_0^{2\pi} \sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}\cos^2\theta\,d\theta = \frac{2E}{\omega}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{2E\pi}{\omega}$

(6) $\Rightarrow E = \frac{J\omega}{2\pi}$

Thus:

(7) $q = \sqrt{\frac{J}{2\pi m}}\sin\theta$ and plugging (4) into (2), $p = \sqrt{2mE}\cos\theta \Rightarrow p = \sqrt{\frac{mJ\omega}{\pi}}\cos\theta$

Now we know $w = \nu t + \beta$, and the usual solution to the oscillator problem is $q = \sqrt{\frac{J}{2\pi m}}\sin(\omega t + \beta)$, $p = \sqrt{\frac{mJ\omega}{\pi}}\cos(\omega t + \beta)$

Thus $\theta = \omega t + \beta$, and since $\omega t + \beta = 2\pi\nu t + \beta$

Define $\beta \equiv \frac{\beta}{2\pi}$, then $\theta = \omega t + \beta = 2\pi(\nu t + \beta) = 2\pi w$

So

$$\boxed{q = \sqrt{\frac{J}{2\pi m}}\sin(2\pi w) \qquad p = \sqrt{\frac{mJ\omega}{\pi}}\cos(2\pi w)}$$