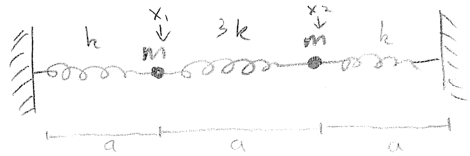
#3
Goldstein 10-18



Find the unshifted Hamiltonian.

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \quad U = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}3k(x_1 - x_2)^2$$

$$L = T - U = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{3}{2}k(x_1 - x_2)^2$$

$P_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$, and similaly $P_{x_2} = m\dot{x}_2$,

$$H = P_{x_1}\dot{x}_1 + P_{x_2}\dot{x}_2 - L \Rightarrow H = T + U \Rightarrow H = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}k(x_1^2 + x_2^2) + \frac{3}{2}k(x_1 - x_2)^2$$

Note – we have a problem. To seperate S, we'll need to seperate H. The problem is the $(x_1 - x_2)^2$ term where we get an ugly cross term.

Point Transformation: $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2 \Rightarrow x_1 = \frac{y_1 + y_2}{2}$, $x_2 = \frac{y_1 - y_2}{2}$

Then $\dot{x}_1 = \frac{\dot{y}_1 + \dot{y}_2}{2}$, $\dot{x}_2 = \frac{\dot{y}_1 - \dot{y}_2}{2}$

and $\dot{x}_1^2 = \frac{1}{4}(y_1^2 + y_1 y_2 + y_2^2)$, $\dot{x}_2^2 = \frac{1}{4}(y_1^2 - y_1 y_2 + y_2^2)$

$$\text{Then } H = \frac{1}{8}m(\dot{y}_1^2 + \cancel{\dot{y}_1\dot{y}_2} + \dot{y}_2^2 + \dot{y}_1^2 - \cancel{\dot{y}_1\dot{y}_2} + \dot{y}_2^2) + \frac{1}{8}k(y_1^2 + \cancel{y_1 y_2} + y_2^2 + y_1^2 - \cancel{y_1 y_2} + y_2^2)$$
$$+ \frac{3}{2}k\left(\cancel{\frac{y_1}{2}} + \frac{y_2}{2} - \cancel{\frac{y_1}{2}} + \frac{y_2}{2}\right)^2$$

$$H = \frac{1}{4}m(\dot{y}_1^2 + \dot{y}_2^2) + \frac{1}{4}k(y_1^2 + y_2^2) + \frac{3}{2}ky_2^2$$

$$\text{and } L = \frac{1}{4}m(\dot{y}_1^2 + \dot{y}_2^2) - \frac{1}{4}k(y_1^2 + y_2^2) - \frac{3}{2}ky_2^2$$

$$P_1 = \frac{\partial L}{\partial \dot{y}_1} = \frac{1}{2}m\dot{y}_1, \quad P_2 = \frac{1}{2}m\dot{y}_2$$

$$H(P_1, P_2, y_1, y_2) = \frac{1}{m}(P_1^2 + P_2^2) + \frac{1}{4}k(y_1^2 + y_2^2) + \frac{3}{2}ky_2^2$$ ← seperable

Hamilton-Jacobi equation

since $P_i = \frac{\partial S}{\partial q_i}$,
$$\frac{1}{m}\left(\left(\frac{\partial S}{\partial y_1}\right)^2 + \left(\frac{\partial S}{\partial y_2}\right)^2\right) + \frac{k}{4}y_1^2 + \frac{7k}{4}y_2^2 = -\frac{\partial S}{\partial t}$$

S is seperable, so $S(y_1, y_2, \alpha_1, \alpha_2, t) = W_1(y_1, \alpha_1) + W_2(y_2, \alpha_2) - \alpha t$

$$\frac{1}{m}\left[\left(\frac{\partial W_1}{\partial y_1}\right)^2 + \left(\frac{\partial W_2}{\partial y_2}\right)^2\right] + \frac{k}{4}y_1^2 + \frac{7k}{4}y_2^2 = \alpha$$

Note! $\alpha = H$ here!

Letting $\alpha_1 + \alpha_2 = \alpha$,

$$\frac{1}{m}\left(\frac{\partial W_1}{\partial y_1}\right)^2 + \frac{k}{4}y_1^2 = \alpha_1 \quad \text{and} \quad \frac{1}{m}\left(\frac{\partial W_2}{\partial y_2}\right)^2 + \frac{7k}{4}y_2^2 = \alpha_2$$

10-18, cont.

rearange to get:

$$\frac{\partial W_1}{\partial y_1} = \sqrt{m\alpha_1 - \frac{km}{4} y_1^2} \qquad \frac{\partial W_2}{\partial y_2} = \sqrt{m\alpha_2 - \frac{7km\, y_2^2}{4}}$$

Use the action vars to get eigenfrequencies.

By 10.102

$$J_1 = \oint \frac{\partial W_1}{\partial y_1} dy_1 \qquad J_2 = \oint \frac{\partial W_2}{\partial y_2} dy_2$$

Another transformation: $y_1 = \sqrt{\frac{4\alpha_1}{K}} \sin(\phi_1)$, $y_2 = \sqrt{\frac{4\alpha_2}{7k}} \sin(\phi_2)$

$dy_1 = \sqrt{\frac{4\alpha_1}{k}} \cos(\phi_1)$

$$J_1 = \sqrt{\frac{4\alpha_1}{k}} \int_0^{2\pi} \sqrt{m\alpha_1(1-\sin^2\phi_1)} \cos\phi_1 \, d\phi_1 = \alpha_1 \sqrt{\frac{4m}{k}} \int_0^{2\pi} \cos^2\phi_1 \, d\phi_1 = \alpha_1 \pi \sqrt{\frac{4m}{k}}$$

$$J_2 = \sqrt{\frac{4\alpha_2}{7k}} \int_0^{2\pi} \sqrt{m\alpha_2(1-\sin^2\phi_2)} \cos\phi \, d\phi_1 = \alpha_2 \sqrt{\frac{4m}{7k}} \int_0^{2\pi} \cos^2\phi_2 \, d\phi_2 = \alpha_2 \pi \sqrt{\frac{4m}{7k}}$$

Thus, $\alpha_1 = \frac{J_1}{\pi}\sqrt{\frac{k}{4m}}$, $\alpha_2 = \frac{J_2}{\pi}\sqrt{\frac{7k}{4m}}$

and $H = \alpha = \alpha_1 + \alpha_2 = \frac{J_1}{\pi}\sqrt{\frac{k}{4m}} + \frac{J_2}{\pi}\sqrt{\frac{7k}{4m}}$

By 10.105, we take $\frac{\partial H}{\partial J_i} = \nu_i$, the eigenfrequencies.

Thus $\boxed{\nu_1 = \sqrt{\frac{k}{4\pi^2 m}}, \quad \nu_2 = \sqrt{\frac{7k}{4\pi^2 m}}}$

10