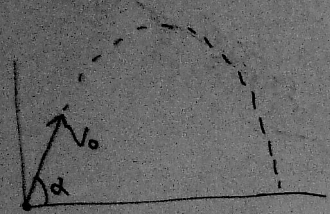
Goldstein 10-17

I worked this out on a whiteboard.

hah! what a good idea

Projectile using H-J



$x(0) = 0, \quad y(0) = 0$

V_0 @ angle α

Get the Hamiltonian.

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad U = mgy$

$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad P_y = m\dot{y}$

$H = \sum_{x,y} \dot{q}_i P_i - L$

$H = m\dot{x}^2 + m\dot{y}^2 - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$

$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$

$H(q, p, t) \rightarrow$ not $\dot{q}$

H-J equation

From 10.2: $H = -\frac{\partial S}{\partial t}$

Also $P_i = \frac{\partial S}{\partial q_i}$

Thus $\frac{\partial S}{\partial x} = m\dot{x}, \quad \frac{\partial S}{\partial y} = m\dot{y}$

and Therefore

$$H = \frac{1}{2}m\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right) + mgy$$

So, the hamilton-jacobi equation becomes:

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy = -\frac{\partial S}{\partial t}$$

Since $S = \frac{1}{2}m(\dot{x} - V_0 \cos\phi) + \left[\frac{1}{2}m(\dot{y} - V_0 \sin\alpha) - mgy\right]$
it is obviously completely seperable.

Thus, we write

$$S(x, y, \alpha_x, \alpha_y, t) = W_x(x, \alpha_x) + W_y(y, \alpha_y) - \alpha t$$ (W's are characteristic fn's.)

Since x is cyclic in the hamiltonian, $W_x = \alpha_x X$

At this point, I guess, we choose to identify $\alpha \equiv \alpha_y$. There can only be n constants α_i
So α has to be one of them. Not entirely sure if we get to choose...

$\therefore \; S = \alpha_x X + W_y(y, \alpha_y) - \alpha_y t$

Plug the principal function into our H-J eqn.:

$$\frac{\alpha_x^2}{2m} + \frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 + mgy = \alpha_y$$

Rearrange: $\left(\frac{\partial W_y}{\partial y}\right)^2 = 2m\alpha_y - \alpha_x^2 - 2m^2gy \rightarrow \frac{\partial W_y}{\partial y} = \sqrt{2m\alpha_y - \alpha_x^2 - 2m^2gy}$

Integrate: $W_y = \int \sqrt{2m\alpha_y - \alpha_x^2 - 2m^2gy}\, dy = \frac{-(2m\alpha_y - \alpha_x^2 - 2m^2gy)^{3/2}}{3m^2g}$

Thus: $S = \alpha_x x - \alpha_y t - \frac{1}{3m^2g}(2m\alpha_y - \alpha_x^2 - 2m^2gy)^{3/2}$

We know that $Q_i = \text{const.} = \beta_i = \frac{\partial S}{\partial \alpha_i}$, so $\beta_x = \frac{\partial S}{\partial \alpha_x}, \; \beta_y = \frac{\partial S}{\partial \alpha_y}$

$\beta_x = x - \frac{\alpha_x}{m^2g}\sqrt{2m\alpha_y - \alpha_x^2 - 2m^2gy} \qquad \beta_y = -t - \frac{1}{mg}\sqrt{2m\alpha_y - \alpha_x^2 - 2m^2gy}$

Solve for $q = q(\alpha, \beta, t)$:

$x(t) = \beta_x + \frac{\alpha_x}{m^2g}\sqrt{2m\alpha_y - \alpha_x^2 - 2m^2gy} \qquad y(t) = \frac{-g}{2}(\beta_y + t)^2 + \frac{\alpha_y}{mg} - \frac{\alpha_x^2}{2m^2g}$

Still must get rid of the y from the x(t) equation. Plug it in...

$$x(t) = \beta_x + \frac{\alpha_x}{m^2g}\sqrt{\cancel{2m\alpha_y} - \cancel{\alpha_x^2} + m^2g^2(\beta_y + t)^2 - \cancel{2m\alpha_y} + \cancel{\alpha_x^2}}$$

$$x(t) = \beta_x + \frac{\alpha_x}{m^2g} mg(\beta_y + t) \rightarrow x(t) = \beta_x + \frac{\alpha_x}{m}\beta_y + \frac{\alpha_x}{m}t$$

continues on back

Now use initial conditions: $x(0)=0,\ y(0)=0,\ \dot{x}(0)=V_0\cos(\alpha)$, $\dot{y}(0)=V_0\sin(\alpha)$

① $\rightarrow x(0)=0 \Rightarrow \beta_x = \frac{\alpha_x}{m}\beta_y \Rightarrow \beta_y = \frac{\beta_x m}{\alpha_x}$

② $\rightarrow y(0)=0 \Rightarrow 0 = \frac{-g}{2}\beta_y^2 + \frac{\alpha_y}{mg} - \frac{\alpha_x^2}{2m^2 g}$

③ $\rightarrow \dot{x}(0) = V_0\cos(\alpha) = \frac{\alpha_x}{m}$

④ $\rightarrow \dot{y}(0) = V_0\sin(\alpha) = -g\beta_y$

4 eqs. 4 unknowns. Solve w/ Mathematica:

$\alpha_x = mV_0\cos(\alpha)$, $\alpha_y = \frac{1}{2}mV_0^2$

$\beta_x = \frac{V_0^2}{g}\cos(\alpha)\sin(\alpha)$, $\beta_y = \frac{-V_0}{g}\sin(\alpha)$

Plug 'em in:

$X(t) = \cancel{\frac{V_0^2}{g}\cos(\alpha)\sin(\alpha)} + \cancel{V_0\cos(\alpha)\left(\frac{-V_0}{g}\sin(\alpha)\right)} + V_0 t\cos(\alpha)$

$$\boxed{X(t) = tV_0\cos(\alpha)}$$

$y(t) = \frac{-g}{2}\left(t - \frac{V_0}{g}\sin(\alpha)\right)^2 + \frac{V_0^2}{2g} - \frac{V_0^2\cos^2(\alpha)}{2g}$

$y(t) = \frac{-g}{2}t^2 + gt\frac{V_0}{g}\sin(\alpha) - \frac{g}{2}\frac{V_0^2}{g^2}\sin^2(\alpha) + \frac{V_0^2}{2g} - \frac{V_0^2\cos^2(\alpha)}{2g}$

$= \frac{-g}{2}t^2 + tV_0\sin(\alpha) - \frac{V_0^2}{2g}\sin^2(\alpha) - \frac{V_0^2}{2g}\cos^2(\alpha) + \frac{V_0^2}{2g}$

$= \frac{-g}{2}t^2 + \frac{V_0^2}{2g} - \frac{V_0^2}{2g} + tV_0\sin(\alpha)$

$$\boxed{y(t) = tV_0\sin(\alpha) - \frac{1}{2}gt^2}$$

As we knew to be correct!

✓

✓

In[27]:=

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Solve[{αx == m v Cos[α], βx == (-αx βy)/m,
  g/2 βy^2 == αy/(m g) - αx^2/(2 m^2 g), v Sin[α] == -g (βy)}, {αx, αy, βx, βy}]
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Out[27]=

$$\left\{\left\{\alpha x \to m\,v\,\mathrm{Cos}[\alpha],\ \alpha y \to \frac{1}{2} m\,v^2\left(\mathrm{Cos}[\alpha]^2 + \mathrm{Sin}[\alpha]^2\right),\ \beta x \to \frac{v^2\,\mathrm{Cos}[\alpha]\,\mathrm{Sin}[\alpha]}{g},\ \beta y \to -\frac{v\,\mathrm{Sin}[\alpha]}{g}\right\}\right\}$$

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