$\boxed{\text{Goldstein 10-16}}$

given $x = \ell(2\phi + \sin(2\phi))$; $y = \ell(1 - \cos(2\phi))$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$; $U = mgy$, $L = T - U$

See the attached Mathematica doc. for the calculation:

$$L = 8m\ell^2\cos^2(\phi)\dot{\phi}^2 - 2mg\ell\sin^2\phi \quad \text{and} \quad P_\phi = \frac{\partial L}{\partial\dot{\phi}} = 16m\ell^2\cos^2(\phi)\dot{\phi}$$

we have the Hamiltonian:

$$H = \dot{q}p - L = 16m\ell^2\cos^2(\phi)\dot{\phi}^2 + 2m\ell\sin^2\phi\,(g - 4\ell\dot{\phi}^2)$$

$$H = E = 8m\ell^2\cos^2(\phi)\dot{\phi}^2 + 2mg\ell\sin^2\phi$$

But we need $H = H(\phi, P_\phi)$, so solve for $\dot{\phi}(P_\phi)$

$$\dot{\phi} = \frac{P_\phi}{16m\ell^2\cos^2\phi} \Rightarrow \dot{\phi}^2 = \frac{P_\phi^2}{256m^2\ell^4\cos^4(\phi)}$$

Thus, $$H = \frac{P_\phi^2}{32m\ell^2\cos^2(\phi)} + 2mg\ell\sin^2(\phi) = E$$ ✓

Since we are looking to find the action variable $J = \oint P_\phi\,d\phi$, we need P_ϕ in terms of constants. So we solve...

$$P_\phi^2 = -64m^2g\ell^3\sin^2\phi\cos^2\phi + 32m\ell^2\cos^2\phi\,E$$

$$P_\phi = 4\ell\sqrt{2mE}\cos\phi\sqrt{1 - \frac{2mg\ell}{E}\sin^2\phi}$$

So now we find J:

$$J = 4\ell\sqrt{2mE}\int_0^{2\pi}\cos(\phi)\sqrt{1 - \frac{2mg\ell}{E}\sin^2\phi}\,d\phi$$

To solve this, use the u-sub:

$$u = \sqrt{\frac{2mg\ell}{E}\sin^2\phi} \Rightarrow du = \cos\phi\sqrt{\frac{2mg\ell}{E}}\,d\phi \Rightarrow d\phi = \frac{1}{\cos\phi}\sqrt{\frac{E}{2mg\ell}}\,du$$

Now, note that since $\phi_{max} = \frac{\pi}{4}$, $u_{max} = \sqrt{\frac{2mg\ell}{E}\sin^2\left(\frac{\pi}{4}\right)}$

✓ Next, notice how $E = \frac{P_\phi^2}{32m\ell^2\cos^2\phi} + 2mg\ell\sin^2(\phi)$, but at ϕ_{max}, $P_\phi = 0$,

so $E = 2mg\ell\sin^2(\phi_{max})$, and $u_{max} = \sqrt{\frac{E}{E}} = 1$

Of course, we need 4 $\frac{\pi}{4}$'s in order to get a full period, so

$$J = \left(\frac{4\ell E}{\sqrt{g\ell}}\int_0^1\sqrt{1-u^2}\,du\right) \times 4 = 16E\sqrt{\frac{\ell}{g}}\int_0^1\sqrt{1-u^2}\,du = 16E\sqrt{\frac{\ell}{g}}\left(\frac{\pi}{4}\right)$$

Thus, $J = 4E\pi\sqrt{\frac{\ell}{g}} \Rightarrow E = \frac{J}{4\pi}\sqrt{\frac{g}{\ell}}$

Since eigenfrequencies are found with $\nu = \frac{\partial H}{\partial J}$, we have $\boxed{\nu = \frac{1}{4\pi}\sqrt{\frac{g}{\ell}}}$ ✓

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In[7]:=
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y[t_] = ι (1 - Cos[2 φ[t]]);
x[t_] = ι (2 φ[t] + Sin[2 φ[t]]);
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In[9]:=
$$L = \frac{1}{2} m \left(\left(x'[t]\right)^2 + \left(y'[t]\right)^2 \right) - m\, g\, y[t] \;//\; \text{FullSimplify}$$

Out[9]=
$$m\, \iota \left(-2\, g\, \text{Sin}[\phi[t]]^2 + 8\, \iota\, \text{Cos}[\phi[t]]^2\, \phi'[t]^2\right)$$