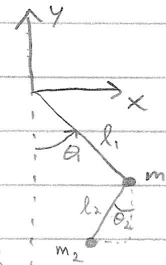
$\boxed{1\text{-}22}$ Here we have a 2-D pendulum w/ masses m_1 and m_2 as well as massles/thin, ridgid "strings" of leangths l_1 & l_2. In cartesian, the positions of m_1 and m_2 are $(x_1, y_1), (x_2, y_2)$ respectively. Here are those coordinates in gen. coods. θ_1, θ_2:



$$\boxed{\begin{aligned}
&x_1 = l_1 \sin\theta_1 \rightarrow \dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 \qquad y_1 = -l\cos\theta_1 \rightarrow \dot{y}_1 = l_1 \sin(\theta_1)\dot{\theta}_1 \\
&x_2 = x_1 - l_2 \sin(\theta_2) = l_1 \sin\theta_1 - l_2 \sin\theta_2 \rightarrow \dot{x}_2 = l_1\cos(\theta_1)\dot{\theta}_1 - l_2 \cos(\theta_2)\dot{\theta}_2 \\
&y_2 = y_1 - l_2\cos(\theta_2) = -l_1\cos(\theta_1) - l_2\cos(\theta_2) \rightarrow \dot{y}_2 = l_1 \sin(\theta_1)\dot{\theta}_1 + l_2\sin(\theta_2)\dot{\theta}_2
\end{aligned}}$$

Then write the kinetic energies:

$$T_1 = \tfrac{1}{2} m_1 (l_1 \cos^2(\theta_1)\dot{\theta}_1^2 + l_2^2 \sin^2(\theta_1)\dot{\theta}_1^2 \Rightarrow T_1 = \tfrac{1}{2} m_1 l_1^2 (\dot{\theta}_1)^2$$

$$\begin{aligned}
T_2 &= \tfrac{1}{2} m_2 (l_1^2 \cos^2(\theta_1)\dot{\theta}_1^2 + l_2^2 \cos^2(\dot{\theta}_2)\dot{\theta}_2^2 + 2 l_1 l_2 \cos(\theta_1)\cos(\theta_2)\dot{\theta}_1\dot{\theta}_2 \\
&\qquad + l_1^2 \sin^2(\theta_1)\dot{\theta}_1^2 + l_2^2 \sin^2(\theta_2)\dot{\theta}_2^2 + 2 l_1 l_2 \sin(\theta_1)\sin(\theta_2)\dot{\theta}_1\dot{\theta}_2) \\
&= \tfrac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)
\end{aligned}$$

$$T_2 = \tfrac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$

Now $T = T_1 + T_2$

$$T = \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] \quad \checkmark$$

we also must write the potential energies:

$$U = U_1 + U_2 = m_1 g y_1 + m_2 g y_2 = -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \quad \checkmark$$

Finally:

$$L = T - U = \tfrac{1}{2} m_2 \left[l_1^2 \dot{\theta}^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] + m_1 g l_1 \cos\theta_1 + m_2 g \left[l_1 \cos\theta_1 + l_2 \cos\theta_2 \right]$$
$$+ \tfrac{1}{2} m l_1^2 \dot{\theta}^2 \quad \checkmark$$

Solution continues on next page

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + (m_1+m_2)gl_1\cos\theta_1 + m_2 g l_2\cos\theta_2$$

Now we have the formidable task of applying the Euler-Lagrange equations twice. First, for θ_1:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = \frac{d}{dt}\left[(m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_2 l_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$$

$$-\left(-m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)gl_1\sin\theta_1\right)$$

$$= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_2 l_1\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_2 l_1\dot{\theta}_2\frac{d}{dt}\left[\cos(\theta_1-\theta_2)\right]$$

$$+ m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + (m_1+m_2)gl_1\sin\theta_1$$

$$= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_2 l_1\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_2 l_1\dot{\theta}_2\dot{\theta}_2\sin(\theta_1-\theta_2)$$

$$\cancel{-m_2 l_2 l_1\dot{\theta}_2\dot{\theta}_1\sin(\theta_1-\theta_2)} \cancel{+m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)} + (m_1+m_2)gl_1\sin\theta_1$$

$$0 = (m_1+m_2)\left[l_1^2\ddot{\theta}_1 + gl_1\sin\theta_1\right] + m_2 l_2 l_1\left[\ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right]$$

$$\boxed{0 = (m_1+m_2)\left[l_1\ddot{\theta}_1 + g\sin\theta_1\right] + m_2 l_2\left[\ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right]}$$ ✓

Next for θ_2:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = \frac{d}{dt}\left[m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2) + m_2 l_2^2\dot{\theta}_2\right]$$

$$-\left(m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2 g l_2\sin(\theta_2)\right)$$

$$= m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) + m_2 l_1 l_2\dot{\theta}_1\left(-\sin(\theta_1-\theta_2)\dot{\theta}_1 + \sin(\theta_1-\theta_2)\dot{\theta}_2\right)$$

$$+ m_2 l_2^2\ddot{\theta}_2 - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + m_2 g l_2\sin(\theta_2)$$

$$= m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2 l_2^2\ddot{\theta}_2 + m_2 g l_2\sin(\theta_2)$$

$$\boxed{0 = m_2 l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2 l_2\ddot{\theta}_2 + m_2 g\sin(\theta_2)}$$ ✓

no physical discussion (forces, momenta, etc)

8/10