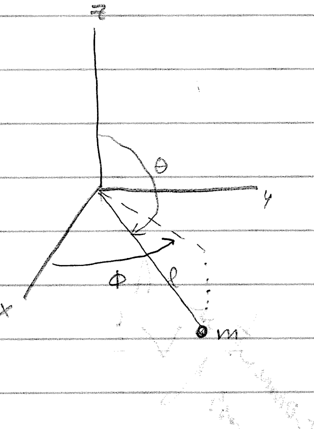
1-19 Here we have the 3-D simple pendulum w/ mass m, and massles/thin, ridgid string l.



in cartesian: $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$, $U=mgz$

We used θ and ϕ as gen. coordinates as defined in diagram →

$$\boxed{\begin{aligned} x&=l\sin\theta\cos\phi &&\rightarrow \dot{x}=-l\sin\theta\,\dot{\phi}\sin\phi+l\dot{\theta}\cos\theta\cos\phi \\ y&=l\sin\theta\sin\phi &&\rightarrow \dot{y}=-l\sin\theta\,\dot{\phi}\cos\phi+l\dot{\theta}\cos\theta\sin\phi \\ z&=l\cos\theta &&\rightarrow \dot{z}=-l\dot{\theta}\sin\dot{\theta}\end{aligned}}$$

Plug these into T:

$$T=\frac{1}{2}m\Big[l^2\sin^2\theta\sin^2\phi(\dot{\phi}^2)+l^2(\dot{\theta}^2)\cos^2\theta\cos^2\phi\cancel{-2l^2\sin\theta\cos\theta\sin\phi\cos\phi(\dot{\phi})(\dot{\theta})}$$
$$+l^2\sin^2\theta\cos^2\phi(\dot{\phi}^2)+l^2\cos^2\theta\sin^2\phi(\dot{\theta}^2)\cancel{+2l^2\sin\theta\cos\theta\sin\phi\cos\phi(\dot{\phi})(\dot{\theta})}$$
$$+l^2\sin^2\theta(\dot{\theta}^2)$$
$$=\frac{1}{2}ml^2\Big[(\dot{\phi}^2)[\sin^2\theta\sin^2\phi+\sin^2\theta\cos^2\phi]+(\dot{\theta}^2)[\cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta]\Big]$$
$$=\frac{1}{2}ml^2\left[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2(\cos^2\theta+\sin^2\theta)\right]=\frac{1}{2}ml^2\left[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right]$$ ✓

also, $U=mgz=mgl\cos\theta$ ✓

1-19 continued

$$L=T-U=\frac{1}{2}ml^2(\dot{\phi}^2\sin^2\theta+\dot{\theta}^2)-mgl\cos\theta$$

Now, we apply the Euler-Lagrange equations. First for θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$$

$$\frac{d}{dt}[ml^2\dot{\theta}]-\left[mgl\sin\theta+\frac{1}{2}ml^2\dot{\phi}^2\sin(2\theta)\right]=0$$

$$\boxed{\ddot{\theta}=\frac{g}{l}\sin\theta+\dot{\phi}^2\sin\theta\cos\theta}$$ ✓

And now for ϕ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\cancel{\frac{\partial L}{\partial\phi}}=0 \Rightarrow \frac{d}{dt}[ml^2\dot{\phi}\sin^2]=0$$

what can you say about this quantity? what is it physically?

$$ml^2\ddot{\phi}\sin^2\theta+2ml^2\dot{\phi}\sin\theta\cos\theta=0$$

$$\ddot{\phi}\sin^2\theta+2\dot{\phi}\sin\theta\cos\theta=0 \Rightarrow \boxed{\ddot{\phi}=-2\dot{\phi}\cot\theta}$$ ✓

no physical discussion

The boxed quantities are the equations of motion.

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