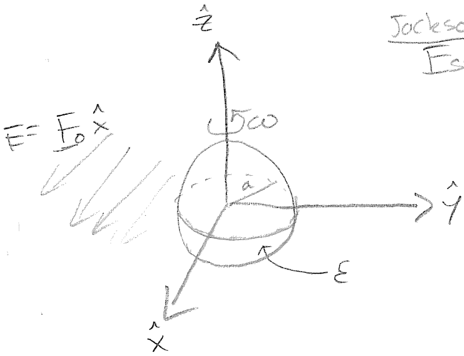

Jackson 6-3)

Essentially, we are asked to find Φ_M.

Begin with equation 4.57: $\vec{P} = 3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\vec{E}_0$

and since $\sigma_{pol} = (\vec{P}\cdot\hat{r})$, we have

$$\sigma_{pol} = 3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)E_0\hat{x}\cdot\hat{r}$$

Since $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$, we have

$$\sigma_{pol} = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi .$$

We'll want this in rectangular form, so use $\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$

Since $\sqrt{x^2+y^2+z^2} = a$ on the sphere (where the polarization is),

so $\sigma_{pol} = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\frac{x}{a}$

The surface current density $\vec{K} = \sigma\vec{v}$

Here we only have bound current rotating around on a sphere's surface,

so $\vec{K} = \sigma_{pol}\cdot\vec{\omega}\times\vec{r} = \sigma_{pol}(\omega\hat{z})\times(a\hat{r}) = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\frac{Ex}{a}\omega a\sin\theta$

$$\vec{K} = 3\varepsilon_0 E_0\omega x\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\frac{\sin\theta}{a}\hat{\phi}$$

By 5.100, $\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int_V\frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \frac{1}{4\pi}\oint_S\frac{\hat{n}'\cdot\vec{M}(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|}$

But in this case, there is no contribution from the volume — just the surface. All we need (besides the inevitable expansion in spherical harmonics!) is $\vec{M}(\vec{x}')$, the magnetization.

$\Rightarrow$ use $\vec{K} = \vec{M}\times\hat{n} = \vec{M}\times\hat{r} \Rightarrow 3\varepsilon_0 E_0\omega x\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\sin\theta\,\hat{\phi} = \vec{M}\times\hat{r}$

$$\vec{M}\times\hat{r} = M\sin\theta\,\hat{\phi} \Rightarrow M = 3\varepsilon_0E_0\omega x\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)$$

and $\vec{M}$'s direction is $\hat{z}$. Thus $\hat{n}'\cdot\vec{M}(\vec{x}') = 3\varepsilon_0E_0\omega x\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\underbrace{\cos\theta}_{\frac{z}{a}}$

$$\Phi_M(\vec{x}) = \frac{3\varepsilon_0E_0\omega}{4\pi a}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\oint_S\frac{x'z'}{|\vec{x}-\vec{x}'|}da' = \frac{3\varepsilon_0E_0\omega}{4\pi a}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\iint\frac{\overbrace{a\sin\theta'\cos\phi'}^{x}\,\overbrace{a\cos\theta'}^{z}\sin\theta'd\theta'd\phi'}{|\vec{x}-\vec{x}'|}$$

Expand in spherical harmonics,

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\Phi_M(\vec{x}) = 3\varepsilon_0E_0\omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\int_0^{2\pi}\int_0^{\pi}\sum_{lm}\frac{\sin^2\theta'\cos\phi'\cos\theta'}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right)Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\,d\theta'd\phi'$$

$$= 3\varepsilon_0E\omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\sum_{lm}\frac{Y_{lm}(\theta,\phi)}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right)\int_0^{2\pi}\int_0^{\pi}\sin^2\theta'\cos\theta'\cos\phi'\,Y^*_{lm}(\theta',\phi')\,d\theta'd\phi'$$

It would be nice to turn $\sin\theta'\cos\theta'$, $\cos\phi'$ into a spherical harmonic (or linear combo) so we can integrate it away to delta functions.

$$Y_{21} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta(\cos\phi + i\sin\phi) \quad \text{and} \quad Y_{2,-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta(\cos\phi - i\sin\phi)$$

$$\text{so } \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(Y_{21} + Y_{2,-1}\right) = \sin\theta\cos\theta\cos\phi$$

$$\Phi_M = 3\varepsilon_0 E_0 \omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\sum_{\ell,m}\frac{Y_{\ell m}(\theta,\phi)}{2\ell+1}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}\right)\int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{2\pi}{15}}\left(Y_{21}+Y_{2-1}\right)Y_{\ell m}^{*}\sin\theta' d\theta' d\phi'$$

The integral becomes:

$$\sqrt{\frac{2\pi}{15}}\left(\delta_{\ell=2,m=1} + \delta_{\ell=2,m=-1}\right)$$

So Φ_M gets some simplification:

$$\Phi_M = 3\varepsilon_0 E_0 \omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\sqrt{\frac{2\pi}{15}}\left[\frac{1}{5}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} + \frac{1}{5}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\phi}\right]\frac{r_<^2}{r_>^3}$$

$$= \frac{3}{10}\varepsilon_0 E_0 \omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\sin\theta\cos\theta(2\cos\phi)\left(\frac{r_<^2}{r_>^3}\right)$$

$$= \frac{3}{5}\varepsilon_0 E_0 \omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\left(\frac{r_<^2}{r_>^3}\right)\frac{xz}{r^2} \qquad \left(\text{b/c} \quad \frac{x}{r} = \sin\theta\cos\phi, \quad \frac{z}{r} = \cos\theta\right)$$

when $r > a$

$$\Phi_m = \frac{3}{5}\varepsilon_0 E_0 \omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\left(\frac{a^2}{r^3}\right)\frac{xz}{r^2} = \frac{3}{5}\varepsilon_0 E_0 \omega\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\frac{a^3}{r^5}xz$$

when $r < a$

$$\Phi_m = \frac{3}{5}\varepsilon_0 E_0 \omega a\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\left(\frac{r^2}{a^3}\right)\frac{xz}{r^2} = \frac{3}{5}\varepsilon_0 E_0 \omega\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\frac{1}{a^2}xz$$

Putting it together

$$\boxed{\Phi_M = \frac{3}{5}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\varepsilon_0 E_0 \omega \cancel{\frac{1}{a^2}}\left(\frac{a}{r_>}\right)^5 xz}$$

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