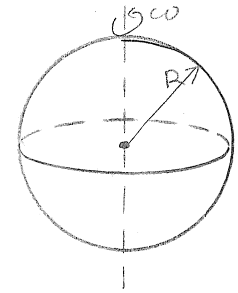
Jackson 6-4



$m = \frac{4}{3}\pi M R^3$

(a) We begin by noticing from 5.105, that for a uniformly magnetized sphere, $H_{in} = -\frac{1}{3}\vec{M}$, $\vec{B}_{in} = \frac{2\mu_0}{3}\vec{M}$.

(I set up my sphere in the same orientation as the one in which those equations were derived - magnetization in $\hat{z}$-direction).

Also the problem says $\vec{m} = \frac{4\pi R^3}{3}\vec{M} \Rightarrow \vec{M} = \frac{3m}{4\pi R^3}\hat{z}$

Thus $\vec{B}_{in} = \frac{\mu_0 m}{2\pi R^3}\hat{z}$

Now we take Ohm's Law: $\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})$

$\Rightarrow \vec{E} = \frac{1}{\sigma}\vec{J} - \vec{v}\times\vec{B}$

Putting ourselves in the rotating frame of the sphere, there is no moving charge, so $J=0$, and

$$\vec{E} = -\vec{V}\times\vec{B} = -(\vec{\omega}\times\vec{r})\times\vec{B} = -\omega B((\hat{z}\times\vec{r})\times\hat{z})$$
$$= -\omega B(\hat{z}\times(\hat{z}\times\vec{r})) = -\omega B(\hat{z}\cdot(\hat{z}\cdot\vec{r}) - \vec{r}(\hat{z}\cdot\hat{z})) = \omega B(\underbrace{\vec{r} - z\hat{z}}_{x\hat{x}+y\hat{y}})$$

✓

$$\Rightarrow \boxed{\vec{E}_{in} = \frac{-\omega\mu_0 m}{2\pi R^3}(x\hat{x}+y\hat{y})}$$

Then we need charge density. We use $\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho$

$$\rho = \varepsilon_0\left(\frac{\partial}{\partial x}E + \frac{\partial}{\partial y}E\right) = \frac{-\cancel{2}\varepsilon_0\mu_0 m\omega}{\cancel{2}\pi R^3}$$

Since $\mu_0\varepsilon_0 = \frac{1}{c^2}$, we have $\boxed{\rho = \text{constant} = -\frac{m\omega}{\pi c^2 R^3}}$ ✓

(b) The multipole expansion is $\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{l,m}\frac{1}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$

with $q_{lm} = \int Y^*_{lm}(\theta',\phi')\,r'^l\rho(\vec{x}')\,d^3r'$ are the multipole moments.

(where $l=0$ is monople, etc.).

We will use our E-field to get $\Phi(\vec{x})$, and compare it with this general expression to extract the moments.

✓

Note that there is no Dipole ~~moment~~ since it would imply a seperation of charge in the $\hat{z}$ direction. We already saw in part a that inside the conductor, E is symmetric, so this can't be the case.

Jackson 6-4, cont.

The potential is $\Phi(\vec{x}) = -\int \vec{E}_{in}\, dl$

Note $x\hat{x} + y\hat{y} = \rho\hat{\rho}$, so $\Phi(\vec{x}) = +\int \frac{\omega \mu_0 m \rho\, d\rho}{2\pi R^3} = V_0 + \frac{\omega \mu_0 m}{4\pi R^3}\rho^2$

But we really are only in cylindrical coords to help the notation — switch back to spherical.

$$\Phi_{in}(\vec{x}) = V_0 + \frac{\omega \mu_0 m}{4\pi R^3} r^2 \sin^2(\theta)$$

Because of spherical symmetry, we expect the potential to be in terms of Legendre Polynomials. (rather than Y_ℓ^m's). Indeed, $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ and $P_0(\cos\theta) = 1$, so $P_0(\cos\theta) - P_2(\cos\theta) = \frac{1}{2}(2 + 1 - 3\cos^2\theta) = \frac{3}{2}(\sin^2\theta)$

$$\Rightarrow \Phi_{in}(\vec{x}) = V_0 + \frac{\omega \mu_0 m}{6\pi R^3} r^2 P_0(\cos\theta) - \frac{\omega \mu_0 m}{6\pi R^3} r^2 P_2(\cos\theta)$$

Cool! at this point we have an $\ell=0$, and $\ell=2$ term. The $\ell=2$ term should probably correspond to the Quadrupole moment, as expected. We also expect to NOT find a monopole term (per problem statement).

Thus the constant $V_0 = \frac{-\omega \mu_0 m}{6\pi R^3} r^2$ (noting $P_0(\cos\theta) = 1$) You won't find it outside or on the surface, but it's there otherwise.

and the potential is just an $\ell = 2$ (quadrupole term).

$$\Phi_{in}(r,\theta) = -\frac{\omega \mu_0 m r^2}{6\pi R^3} P_2(\cos\theta) \quad \text{in the sphere.}$$

We need potential outside the sphere. We do this be realizing that $\frac{r^2}{R^3}$ is really jus $\frac{r_<^2}{r_>^3}$ which becomes $\frac{R^2}{r^3}$

$$\Phi_{out}(r,\theta) = \frac{-\omega \mu_0 R^2 m}{6\pi r^3} P_2(\cos\theta)$$

Compare this with the multipole expansion (first line of (b), previous page).

$P_2(\cos\theta) = 2\sqrt{\frac{\pi}{5}}\, Y_2^0(\theta,\phi)$ and setting $\ell = 2$, we get

$$\Phi_{multipole} = \frac{1}{\varepsilon_0}\frac{1}{5} q_{20} \frac{1}{2}\sqrt{\frac{5}{\pi}} \frac{P_2(\cos\theta)}{r^3} = \frac{1}{\varepsilon_0} q_{20} \sqrt{\frac{1}{20\pi}}\, P_2(\cos\theta) \frac{1}{r^3}$$

equating with Φ_{out} we get $q_{20} = \frac{-\omega R^2 m \sqrt{20\pi}}{6c^2\pi} = \frac{-\omega R^2 m}{3c^2}\sqrt{\frac{5}{\pi}}$

By 4.6, $Q_{33} = 2\sqrt{\frac{4\pi}{5}}\, q_{20} \Rightarrow \boxed{Q_{33} = -\frac{4\omega R^2 m}{3c^2}}$ ✓

Also, Q_{ij} is a traceless tensor of rank 2, so $Q_{11} + Q_{22} = -Q_{33}$ ✓ and symmetry tells us $Q_{11} = Q_{22}$, so $\boxed{Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}}$

(c) For finding $\sigma(\theta)$, we use the Boundary Condition

$$(\vec{E}_{out}|_{r=R} - \vec{E}_{in}|_{r=R})\hat{n} = \sigma/\varepsilon_0 \qquad \text{(in a vacuum).}$$

from pt. (c), $\vec{E}_{out} = -\vec{\nabla}\Phi_{out} = \frac{d}{dr}\left(\frac{\omega \mu_0 R^2 m}{6\pi r^3} P_2(\cos\theta)\right)\hat{r} + \frac{d}{d\theta}\left(\frac{\omega \mu_0 R^2 m}{6\pi r^4} P_2(\cos\theta)\right)$

$$\vec{E}_{OUT} = \frac{-\omega \mu_0 R^2 m}{2\pi r^4} P_2(\cos\theta)\hat{r} - \frac{\omega \mu_0 R^2 m}{2\pi r^4}\cos\theta\sin\theta\,\hat{\theta}$$

similarly

$$\vec{E}_{in} = \frac{\omega \mu_0 m r}{3\pi R^3}[1 - P_2(\cos\theta)]\hat{r} - \frac{\omega \mu_0 m r^2}{2\pi R^3}\cos\theta\sin\theta\,\hat{\theta}$$

noting $\hat{n} = \hat{r}$ in this case, the B.C. gives us

$$\sigma/\varepsilon_0 = \frac{\omega \mu_0 m}{3\pi R^2}\left[1 - P_2(\cos\theta)\right] - \frac{\omega \mu_0 m}{2\pi R^2} P_2(\cos\theta)$$

$$\sigma(\theta) = \frac{\omega m}{\pi R^2} P_2(\cos\theta)\left(-\frac{1}{3} - \frac{1}{2}\right) + \frac{\omega m}{3\pi R^2 c^2}$$

$$\sigma(\theta) = \frac{\omega m}{\pi R^2} P_2(\cos\theta) - \frac{5}{6}\frac{\omega m}{\pi R^2} P_2(\cos\theta)$$

Simplifying to look like Jackson...

$$\boxed{\sigma(\theta) = \frac{1}{4\pi R^2}\cdot\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]} \checkmark$$

(d) Let's choose a nice, simple path. We integrate the $\vec{E}_{in}\cdot\hat{\theta}|_{r=R}$ from $\theta = \pi/2 \rightarrow 0$ **(Why are all paths equivalent here?)**

$$\mathcal{E} = \int_{\pi/2}^{0} \frac{-\omega \mu_0 m}{2\pi R}\cos\theta\sin\theta\, d\theta = \boxed{\frac{+\omega \mu_0 m}{4\pi R}} \checkmark$$

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