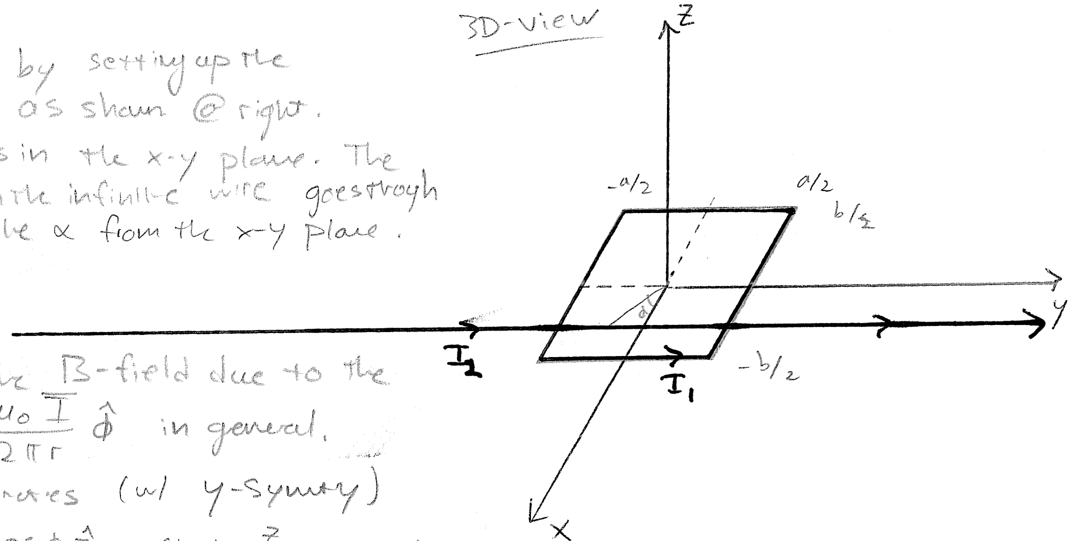
#6 Jackson 5-25

(a) We begin by setting up the problem as shown @ right. The loop is in the x-y plane. The current in the infinite wire goes through at an angle $\alpha$ from the x-y plane.



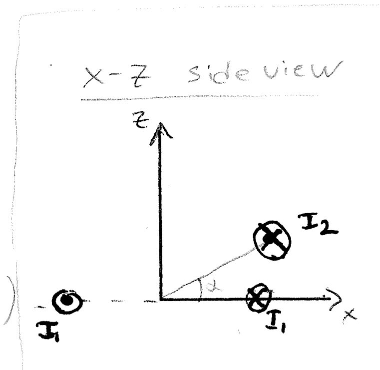

First, let's find the B-field due to the long wire. $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$ in general.

In cartesian coordinates (w/ y-symmetry)

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{z}$, $\sin\phi = \frac{z}{r}$, $\cos\phi = \frac{x}{r}$,

and $r = \sqrt{x^2+z^2}$. Plugging it all in,

$$\vec{B} = \frac{-\mu_0 I z\,\hat{x}}{2\pi(x^2+z^2)} + \frac{\mu_0 I x\,\hat{z}}{2\pi(x^2+z^2)}$$




In our case, $z \to z - d\sin\alpha$, $x \to x - d\cos\alpha$, and $I \to I_2$

$$\vec{B} = \frac{\mu_0 I_2}{2\pi\left[(x-d\cos\alpha)^2 + (z-d\sin\alpha)^2\right]}\left(-(z-d\sin\alpha)\hat{x} + (x-d\cos\alpha)\hat{z}\right)$$

Now we are given $W_{12} = I_1 F_2 = I_1 \int_S \vec{B}\cdot(-\hat{z})\,da = I_1 \left.\left(\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} dy\,dx\, \frac{\mu_0 I_2 (x - d\cos\alpha)}{2\pi\left[(x-d\cos\alpha)^2 + (z-d\sin\alpha)^2\right]}\right)\right|_{z=0}$

(we set $z=0$ b/c all the interaction Magnetic Energy happens @ $z=0$ (@ the loop)).

$$W_{12} = \frac{\mu_0 I_1 I_2 a}{2\pi}\left.\int_{-b/2}^{b/2} \frac{-(x-d\cos\alpha)}{(x-d\cos\alpha)^2 + (z-d\sin\alpha)^2}\,dx\right|_{z=0}$$

$$= \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\left.\left(\frac{\frac{b^2}{4} + d^2 + \cancel{z^2} + bd\cos\alpha - \cancel{2dz\sin\alpha}}{\frac{b^2}{4} + d^2 + \cancel{z^2} - bd\cos\alpha - \cancel{2dz\sin\alpha}}\right)\right|_{z=0}$$

$$\boxed{W_{12} = \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\left(\frac{b^2 + 4d^2 + 4bd\cos\alpha}{b^2 + 4d^2 - 4bd\cos\alpha}\right)}$$ ✓

(b) $\vec{F} = \overset{\text{side length}}{l}\,\vec{I}\times\vec{B}$. In this case the sides of length b (parallel to $\hat{x}$) feel no net force since the B-field is y-independent, and these sides' contributions cancel via symmetry. We need to calculate $\vec{B}$ at $z=0$, $x = b/2$ and $z=0$, $x=-b/2$

$$\vec{B}(-b/2, 0) = \frac{\mu_0 I_2}{2\pi\left(-\frac{b}{2} - d\cos\alpha\right)^2 + d^2\sin^2\alpha}\left[(d\sin\alpha)\hat{x} + \left(-\frac{b}{2} - d\cos\alpha\right)\hat{z}\right]$$

$$\vec{B}(b/2, 0) = \frac{\mu_0 I_2}{2\pi\left(\frac{b}{2} - d\cos\alpha\right)^2 + d^2\sin^2\alpha}\left[(d\sin\alpha)\hat{x} + \left(\frac{b}{2} - d\cos\alpha\right)\hat{z}\right]$$

since $\vec{u}\times\vec{v} = (u_y v_z - u_z - v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z}$,

and $\vec{F} = I(\vec{L}\times\vec{B}) = I_1(a\hat{y}\times\vec{B})$ in this case, we get

$$\vec{F} = (aB_z)\hat{x} + (-aB_x)\hat{z} \quad \text{or} \quad F_x = aB_z \quad \text{and} \quad F_z = -aB_x$$

where we evaluate $B_z$ and $B_x$ on the two stretches of wire at $x=\pm\frac{b}{2}$, $z=0$. Each force has a contribution from the two sides.

$$\boxed{\begin{aligned} F_x &= I_1\left(aB_z(\tfrac{b}{2},0) - aB_z(\tfrac{-b}{2},0)\right) = I_1 a\left(\frac{\mu_0 I_2(\frac{b}{2}-d\cos\alpha)}{2\pi(\frac{b}{2}-d\cos\alpha)^2+d\sin^2\alpha} - \frac{\mu_0 I_2(\frac{-b}{2}-d\cos\alpha)}{2\pi(\frac{-b}{2}-d\cos\alpha)^2+d\sin^2\alpha}\right) \\ F_z &= -I_1\left(aB_x(\tfrac{b}{2},0) - aB_x(\tfrac{-b}{2},0)\right) = -I_1 a\left(\frac{\mu_0 I_2 d\sin\alpha}{2\pi(\frac{b}{2}-d\cos\alpha)^2+d\sin^2\alpha} - \frac{\mu_0 I_2 d\sin\alpha}{2\pi(\frac{-b}{2}-d\cos\alpha)^2+d\sin^2\alpha}\right)\end{aligned}}$$

✓

(c) $W_{12} = \int \vec{J}_1\cdot\vec{A}_2\, d^3x = I_1\oint d\vec{l}_1\cdot\vec{A}_2$ when, in this case the current $I_1$ is constant around the loop.

From 5.32, $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{I_2\, dl_2}{r} \Rightarrow \vec{A}_2(\vec{x}) = \frac{-I_2\mu_0}{4\pi}\ln(x^2+z^2)\hat{y}$ for a y-oriented infinite wire through the origin. In this case, we shift the wire such that

$$\vec{A}_2 = \frac{-I_2\mu_0}{4\pi}\ln\left((x-d\cos\alpha)^2+(z-d\sin\alpha)^2\right)\hat{y}$$

In the cylindrical case, $x\to a\cos\phi$, and $dl_y\to a\cos\phi\, d\phi$, so

$$W_{12} = I_1\int_0^{2\pi}\frac{-I_2\mu_0}{4\pi}\ln\left((a\cos\phi-d\cos\alpha)^2+d^2\sin^2\alpha\right)a\cos\phi\, d\phi$$

$$= \frac{-I_2 I_1\mu_0}{4\pi}\int_0^{2\pi}\ln\left(a^2\cos^2\phi - 2da\cos\phi\cos\alpha + 1\right)a\cos\phi\, d\phi$$

Then somehow integrate this awful thing

(d) ... sorry ^^

This problem is nasty, so points for trying each part!

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The integral from 5.25a . . .



$$\int_{-b/2}^{b/2} \frac{(x - d\,\mathrm{Cos}[\alpha])}{(x - d\,\mathrm{Cos}[\alpha])^2 + (z - d\,\mathrm{Sin}[\alpha])^2}\,dx$$

ConditionalExpression[

$$\frac{1}{2}\left(\mathrm{Log}\left[\frac{b^2}{4} + d^2 + z^2 - b\,d\,\mathrm{Cos}[\alpha] - 2\,d\,z\,\mathrm{Sin}[\alpha]\right] - \mathrm{Log}\left[\frac{b^2}{4} + d^2 + z^2 + b\,d\,\mathrm{Cos}[\alpha] - 2\,d\,z\,\mathrm{Sin}[\alpha]\right]\right),$$

$$\left(\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \neq\right.$$

$$\mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \,||\, \mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] -$$

$$\mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] - \mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \geq$$

$$\frac{1}{2} \,||\, -\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\left.\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \geq \frac{1}{2}\right) \&\&$$

$$\left(\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \neq\right.$$

$$\mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \,||$$

$$-\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] -$$

$$\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \geq \frac{1}{2} \,||$$

$$\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] - \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\left.\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \geq \frac{1}{2}\right) \&\&$$

$$\left(\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \neq\right.$$

$$\mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \,||$$

$$\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(-d\,\mathrm{Cos}[\tfrac{\alpha}{2}] + z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \neq$$

$$\mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \,||$$

$$\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] - \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] -$$



$$\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \geq \frac{1}{2} \,||$$

$$\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] - \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\left.\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(-d\,\mathrm{Cos}[\tfrac{\alpha}{2}] + z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] < -\frac{1}{2}\right) \&\&$$

$$\left(\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \neq\right.$$

$$\mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \,||$$

$$\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(-d\,\mathrm{Cos}[\tfrac{\alpha}{2}] + z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \neq$$

$$\mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \,||$$

$$-\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] < -\frac{1}{2} \,||$$

$$\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] - \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\left.\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] - i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(-d\,\mathrm{Cos}[\tfrac{\alpha}{2}] + z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] < -\frac{1}{2}\right) \&\&$$

$$\left(\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \neq\right.$$

$$\mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \,||$$

$$\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] - \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(d\,\mathrm{Cos}[\tfrac{\alpha}{2}] - z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] < -\frac{1}{2} \,||$$

$$\mathrm{Im}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(-d\,\mathrm{Cos}[\tfrac{\alpha}{2}] + z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Re}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] \neq$$

$$\mathrm{Re}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] \,||$$

$$-\mathrm{Im}\left[\frac{z\,\mathrm{Cos}[\tfrac{\alpha}{2}]\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] + \mathrm{Im}\left[\frac{d\,(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,\mathrm{Sin}[\tfrac{\alpha}{2}]}{b}\right] +$$

$$\left.\mathrm{Re}\left[\frac{(\mathrm{Cos}[\tfrac{\alpha}{2}] + i\,\mathrm{Sin}[\tfrac{\alpha}{2}])\,(-d\,\mathrm{Cos}[\tfrac{\alpha}{2}] + z\,\mathrm{Sin}[\tfrac{\alpha}{2}])}{b}\right] < -\frac{1}{2}\right)]$$