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Jackson 5-16

(a) First off, note that the problem in the absence of iron is worked out in Section 5-5 of Jackson. John David Jackson gets:

$$B_r = \frac{\mu_0 I a}{2r} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}(\cos\theta)$$

$$B_\theta = -\frac{\mu_0 I a^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \begin{Bmatrix} -\left(\frac{2n+2}{2n+1}\right)\frac{1}{a^3}\left(\frac{r}{a}\right)^{2n} & \leftarrow r<a \\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2n} & \leftarrow r>a \end{Bmatrix} P'_{2n+1}(\cos\theta)$$

Second, we have a contribution to the B-field from the iron. With beautiful spherical symmetry, we know it takes the form

$$\Phi_M = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Call this contribution to the B-field B'. $\vec{B}' = -\vec{\nabla}\Phi_M$

$$\vec{B}' = \frac{-\partial \Phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \Phi_M}{\partial\theta}\hat{\theta}$$

so $B'_r = -\sum_{l=0} l A_l r^{l-1} P_l(\cos\theta)$, $-B_\theta' = \sum_l A_l r^{l-1} P'_l(\cos\theta)$

At the Boundary, Tangential components are equal, so

$$\sum A_l b^{l-1} P'_l(\cos\theta) = \frac{\mu_0 I a^2}{4} \sum_l \frac{(-1)^l (2l+1)!!}{2^l (l+1)!} \frac{1}{b^3}\left(\frac{a}{b}\right)^{2l} P'_{2l+1}(\cos\theta)$$

Taking each individual term:

$$\underbrace{A_l P'_l(\cos\theta)} = \frac{\mu_0 I a^2}{4b^3} \frac{(-1)^l (2l+1)!!}{2^l (l+1)!} \left(\frac{a^{2l}}{b^{3l-1}}\right) P'_{2l+1}(\cos\theta)$$

and plug into B'_r and B'_θ

$$B'_r = \frac{\mu_0 I a^2}{4b^3} \sum_{l=0}^{\infty} \frac{l r^{l-1} (-1)^l (2l+1)!!}{2^l (l+1)!} \left(\frac{a^{2l}}{b^{3l-1}}\right) P_{2l+1}(\cos\theta)$$

so

$$B'_r|_{r\to 0} = \frac{\mu_0 I a^2}{4b^3}$$

Now we also have $B_r|_{r\to 0} = \frac{\mu_0 I a}{2r}\left(\frac{r^{2n+1}}{a^{2n+2}}\right) \xrightarrow{n=0 \text{ dominates}} \frac{\mu_0 I a}{2r}\frac{r}{a^2} = \frac{\mu_0 I}{2a}$

Then $B_{r\,(tot)} = \frac{\mu_0 I a^2}{4b^3} + \frac{\mu_0 I}{2a} = \boxed{\frac{\mu_0 I}{2a}\left(1 + \frac{a^3}{2b^3}\right)}$ ✓

$$\boxed{B_{r\,(tot)} = B_r\left(1 + \frac{a^3}{2b^3}\right)}$$ ✓

(b) We can think about a larger current loop of radius R. Then $B_z = \frac{\mu_0 I}{2R}$ right at the center. (This is dipole aprox.)

But if we introduce the iron from part a, we evidently get an "augmentation" factor: $\left(1 + \frac{a^3}{2b^3}\right)$.

so $$B_{r,tot} = \frac{\mu_0 I}{2a}\left(1 + \frac{a^3}{2b^3}\right) = \frac{\mu_0 I}{2a} + \frac{\mu_0 I a^3}{2(a\,2b^3)}$$

$$= \frac{\mu_0 I}{2a} + \frac{\mu_0 I}{2\left(2b^3/a^2\right)}$$

Thus, the B-field is composed of a "real loop" of radius a, and an image with contribution $\frac{\mu_0 I}{2\left(2b^3/a^2\right)}$,

and this term has radius $\boxed{\frac{2b^3}{a^2}}$ ✓

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