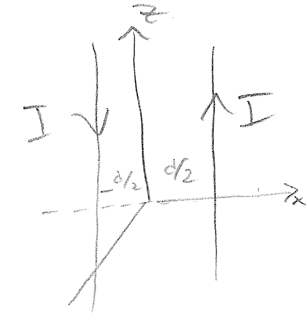
#4
Jackson 5.15



⓪ Let's start by thinking of just one current-carrying wire flowing up thru x,y=0. Then we'll shift it right to $x = d/2$, and add a second wire in at $x = -d/2$ by superposition.

→ Draw a small ampèrian loop around our single wire (radius ρ)

$\oint \vec{H} \cdot d\vec{l} = I_{enc} = I$ ← Ampère's Law

$\vec{H}$ and $d\vec{l}$ are both entirely in the $\hat{\phi}$ direction, so we can drop the integral.

$H \cdot 2\pi a \hat{\phi} = I \Rightarrow H = \frac{I}{2\pi\rho}\hat{\phi}$

→ Given the definition $\vec{H} = -\vec{\nabla}\Phi_M$, we have $\vec{\nabla}\Phi_M = \frac{-I}{2\pi\rho}\hat{\phi}$

in cylindrical: $\underbrace{\cancel{\frac{\partial \Phi_M}{\partial \rho}\hat{\rho}}}_{\text{infinite = no } \rho \text{ dependence.}} + \frac{1}{\rho}\frac{\partial \Phi_M}{\partial \phi}\hat{\phi} + \underbrace{\cancel{\frac{\partial \Phi_M}{\partial z}\hat{z}}}_{\text{clearly symmetric = no } z \text{ dependence}} = \frac{-I}{2\pi\rho}\hat{\phi}$

$\frac{1}{\rho}\frac{\partial \Phi_M}{\partial \phi}\hat{\phi} = \frac{-I}{2\pi\rho}\hat{\phi} \Rightarrow d\Phi_M = \frac{-I}{2\pi}d\phi \longrightarrow \text{Integrate} \longrightarrow \Phi_M = \frac{-I}{2\pi}\phi$

→ Plug in $\tan(\phi) = \frac{y}{x} \Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right) = \phi \Rightarrow \Phi_M = \frac{-I}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$

→ Now shift that current-carrying wire to $x = +\frac{d}{2} \Rightarrow \Phi_M = \frac{-I}{2\pi}\tan^{-1}\left(\frac{y}{x - d/2}\right)$

and add in the other wire: $\Phi_M = \frac{I}{2\pi}\left(\tan^{-1}\left(\frac{y}{x + d/2}\right) - \tan^{-1}\left(\frac{y}{x - d/2}\right)\right)$.

Then go back to cylindrical coordinates, (with $x = \rho\cos\phi$, $y = \rho\sin\phi$)

$$\Phi_M = \frac{I}{2\pi}\left(\tan^{-1}\left(\frac{\rho\sin\phi}{\rho\cos\phi + d/2}\right) - \tan^{-1}\left(\frac{\rho\sin\phi}{\rho\cos\phi - d/2}\right)\right)$$

Since $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$, plug in $u = \tan\alpha$, $V = \tan\beta$

and get $\tan^{-1}(u) + \tan^{-1}(V) = \tan^{-1}\left(\frac{u + V}{1 - uV}\right)$

so

$$\Phi_M = \frac{-I}{2\pi}\tan^{-1}\left(\frac{\frac{\rho\sin\phi}{\rho\cos\phi + \frac{d}{2}} - \frac{\rho\sin\phi}{\rho\cos\phi - d/2}}{1 + \frac{\rho^2\sin^2\phi}{\rho^2\cos^2\phi - d^2/4}}\right) = \frac{-I}{2\pi}\tan^{-1}\left(\frac{4d\rho\sin\phi}{d^2 - 4\rho^2}\right)$$

$$\Phi_M = \frac{-I}{2\pi}\tan^{-1}\left(\frac{\left(\frac{\rho}{d}\right)\sin\phi}{\left(\frac{d}{\rho}\right)^2\left(\frac{-1}{4}\right) + 1}\right) \overset{\text{taylor}}{\approx} \frac{-I}{2\pi}\left[\left(\frac{\left(\frac{d}{\rho}\right)\sin\phi}{-\frac{1}{4}\left(\frac{d}{\rho}\right)^2 + 1}\right) - \frac{\frac{\left(\frac{\rho}{d}\right)^3\sin^3\phi}{\left(-\frac{1}{4}\left(\frac{d}{\rho}\right)^2 + 1\right)^3}}{3} + - + \cdots\right]$$

In the limit of small spacing, $d \ll \rho$, so powers of $\frac{d}{\rho}$ greater than 1 become irrelevant. Ignore those terms:

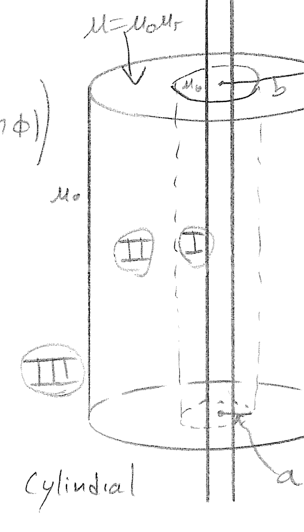
$$\boxed{\Phi_M \approx -\frac{Id\sin\phi}{2\pi\rho}} \checkmark$$

(b) In general in Polar Coordinates:

$$\Phi = \sum_{m=0}^{\infty} (\alpha_m \rho^{m} + \beta_m \rho^{-m})(A_m\cos(m\phi) + B_m \sin(m\phi))$$

We get rid of the cos terms in order to match w/ Φ_M.

Define regions I, II, and III as shown →



Then by orthogonality only m=1 contributes.

we will consider the total potential in each region as the linear superposition of the "free space" Φ_M and the Cylindrical Object's contribution.

$$\Phi = \Phi_M + \left(\alpha\rho + \frac{\beta}{\rho}\right)(B\sin(m\phi))$$

(I) in the limit $\rho \to 0$, Φ_I should not explode. Thus $\beta = 0$

$$\Phi_I = \Phi_M + \gamma \sin\phi\, \rho = \frac{-Id\sin\phi}{2\pi}\left(\frac{1}{\rho} - \gamma\rho\right)$$

(II) $$\Phi_{II} = \Phi_M + \left(\alpha\rho + \frac{\beta}{\rho}\right)B(\sin\phi) = \frac{-Id\sin\phi}{2\pi}\left(\delta\rho + \epsilon\frac{1}{\rho}\right)$$

(III) in the limit $\rho \to \infty$, Φ_{III} shouldn't explode. Thus $\alpha = 0$.

$$\Phi_{III} = \Phi_M + \frac{\eta\sin\phi}{\rho} = \frac{-Id\sin\phi}{2\pi}\left(\eta/\rho\right)$$

So now we have 4 unknowns. We need Boundary Conditions to lock them down. ($H^{\parallel}_{(+)} = H^{\parallel}_{(-)}$, $B^{\perp}_{(+)} = B^{\perp}_{(-)}$). So, lets find $\vec{H}$.

$$\vec{H} = -\nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat\phi + \cancel{\frac{\partial\Phi}{\partial z}\hat z}$$ ← independent of z.

$$\vec{H}_I = \frac{+Id}{2\pi}\left[\left(-\gamma - \frac{1}{\rho^2}\right)\sin\phi\,\hat\rho + \left(\frac{1}{\rho^2} - \gamma\right)\cos\phi\,\hat\phi\right]$$

$$\vec{H}_{II} = \frac{+Id}{2\pi}\left[\left(\delta - \frac{\epsilon}{\rho^2}\right)\sin\phi\,\hat\rho + \left(\frac{\epsilon}{\rho^2} + \delta\right)\cos\phi\,\hat\phi\right]$$

$$\vec{H}_{III} = \frac{-Id\eta}{2\pi\rho^2}\left[\sin\phi\,\hat\rho - \cos\phi\,\hat\phi\right]$$

Now apply Boundary Conditions. $\vec{H}_I\cdot\hat\phi\big|_a = \vec{H}_{II}\cdot\hat\phi\big|_a \Rightarrow \frac{1}{a^2} - \gamma = \frac{\epsilon}{a^2} + \delta$

$\vec{H}_{II}\cdot\hat\phi\big|_b = \vec{H}_{III}\cdot\hat\phi\big|_b \Rightarrow \left(\frac{\epsilon}{b^2} + \delta\right) = \frac{\eta}{b^2}$;

$\vec{B}_I\cdot\hat\rho\big|_a = \vec{B}_{II}\cdot\hat\rho\big|_a \Rightarrow \vec{H}_I\cdot\hat\rho\big|_a = \mu_r\vec{H}_{II}\cdot\hat\rho\big|_a \Rightarrow \left(-\gamma - \frac{1}{a^2}\right) = \mu_r\left(\delta - \frac{\epsilon}{a^2}\right)$

final B.C. $\vec{B}_{II}\cdot\hat{\rho}\big|_b = \vec{B}_{III}\cdot\hat{\rho}\big|_b \Rightarrow \mu_r \vec{H}_{II}\cdot\hat{\rho}\big|_b = \vec{H}_{III}\cdot\hat{\rho}\big|_b \Rightarrow \mu_r\left(\delta - \frac{\varepsilon}{a^2}\right) = \frac{\eta}{a^2}$

Now solve the few squigly underlined equations for $\gamma, \delta, \varepsilon, \eta$ (in mathematica): Then plug them in

$$\Phi_I = \frac{-Id\sin\phi}{2\pi}\left(\frac{1}{\rho} - \frac{\rho}{a^2}\,\frac{(\mu_r^2-1)(a^2-b^2)}{a^2(\mu_r-1)^2 - b^2(\mu_r+1)^2}\right)$$

$$\Phi_{II} = \frac{-Id\sin\phi}{2\pi}\left(\frac{2(1-\mu_r)\,\rho}{a^2(\mu_r-1)^2 - b^2(\mu_r+1)^2} + \frac{2b^2(\mu_r+1)}{\rho\, b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}\right)$$

$$\Phi_{III} = \frac{-Id\sin\phi}{2\pi\rho}\left(\frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}\right) = \frac{-Id\sin\phi}{2\pi\rho}F = \boxed{\Phi_M F}$$

Where F is defined in the problem statment.

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Electrodynamics Homework 5 — Ben Levy

#4 Jackson Problem 5.15b

These are the general expressions for the potential in the 3 regions. I will need to find the constants γ,δ,ϵ, and η.

In[73]:=
$$\Phi 1[\rho\_] = \frac{-\mathrm{I}\,\mathrm{d}\,\mathrm{Sin}[\phi]}{2\,\pi}\left(\frac{1}{\rho}-\gamma\,\rho\right);$$
$$\Phi 2[\rho\_] = \frac{-\mathrm{I}\,\mathrm{d}\,\mathrm{Sin}[\phi]}{2\,\pi}\left(\delta\,\rho+\epsilon\,\frac{1}{\rho}\right);$$
$$\Phi 3[\rho\_] = \frac{-\mathrm{I}\,\mathrm{d}\,\mathrm{Sin}[\phi]}{2\,\pi}\left(\frac{\eta}{\rho}\right);$$

The following are the ρ and ϕ components of the H-field in each of the three regions, written as functions of ρ.

In[92]:=
$$\{\mathrm{H1}\rho[\rho\_] = \mathrm{D}[\Phi 1[\rho], \rho],\ \mathrm{H1}\phi[\rho\_] = \frac{1}{\rho}\mathrm{D}[\Phi 1[\rho], \phi]\}$$

Out[92]=
$$\left\{-\frac{\mathrm{d}\,\mathrm{I}\left(-\gamma-\frac{1}{\rho^2}\right)\mathrm{Sin}[\phi]}{2\,\pi},\ -\frac{\mathrm{d}\,\mathrm{I}\left(\frac{1}{\rho}-\gamma\,\rho\right)\mathrm{Cos}[\phi]}{2\,\pi\,\rho}\right\}$$

In[94]:=
$$\{\mathrm{H2}\rho[\rho\_] = \mathrm{D}[\Phi 2[\rho], \rho],\ \mathrm{H2}\phi[\rho\_] = \frac{1}{\rho}\mathrm{D}[\Phi 2[\rho], \phi]\}$$

Out[94]=
$$\left\{-\frac{\mathrm{d}\,\mathrm{I}\left(\delta-\frac{\epsilon}{\rho^2}\right)\mathrm{Sin}[\phi]}{2\,\pi},\ -\frac{\mathrm{d}\,\mathrm{I}\left(\frac{\epsilon}{\rho}+\delta\,\rho\right)\mathrm{Cos}[\phi]}{2\,\pi\,\rho}\right\}$$

In[95]:=
$$\{\mathrm{H3}\rho[\rho\_] = \mathrm{D}[\Phi 3[\rho], \rho],\ \mathrm{H3}\phi[\rho\_] = \frac{1}{\rho}\mathrm{D}[\Phi 3[\rho], \phi]\}$$

Out[95]=
$$\left\{\frac{\mathrm{d}\,\eta\,\mathrm{I}\,\mathrm{Sin}[\phi]}{2\,\pi\,\rho^2},\ -\frac{\mathrm{d}\,\eta\,\mathrm{I}\,\mathrm{Cos}[\phi]}{2\,\pi\,\rho^2}\right\}$$

The following solves the system of equations in which I have equated the tangential (ϕ) components of the H-field at the two boundaries, and the normal (ρ) components of the B-field at the boundaries.

In[97]:=
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Solve[{H1ϕ[a] == H2ϕ[a], H2ϕ[b] == H3ϕ[b], H1ρ[a] == μr * H2ρ[a], μr * H2ρ[b] == H3ρ[b]},
  {γ, δ, ϵ, η}] // FullSimplify
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Out[97]=
$$\left\{\left\{\gamma\to-\frac{(a-b)\,(a+b)\,\left(-1+\mu_r^2\right)}{a^2\,(a+b+(-a+b)\,\mu_r)\,(-a+b+(a+b)\,\mu_r)},\right.\right.$$
$$\delta\to\frac{2\,(-1+\mu_r)}{(a+b+(-a+b)\,\mu_r)\,(-a+b+(a+b)\,\mu_r)},\ \epsilon\to b\left(\frac{1}{a+b+(-a+b)\,\mu_r}+\frac{1}{-a+b+(a+b)\,\mu_r}\right),$$
$$\left.\left.\eta\to\frac{4\,b^2\,\mu_r}{(a+b+(-a+b)\,\mu_r)\,(-a+b+(a+b)\,\mu_r)}\right\}\right\}$$