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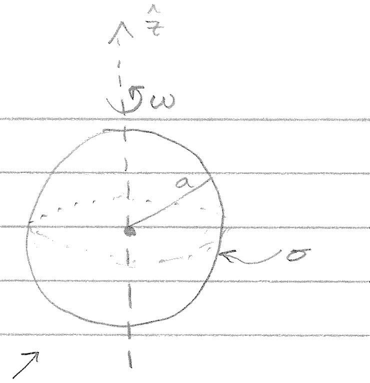
Jacksen 5-13

We start by writing the charge distribution

$\vec{J} = \sigma \vec{v}\, \delta(r'-a)\,\hat{\phi}'$ and $\vec{v} = \vec{\omega}\times\vec{r} = \omega r \sin\theta\, \hat{v}$

so $\vec{J}(\vec{x}') = \sigma \omega a \sin\theta'\, \delta(r'-a)\,\hat{\phi}'$



Find $\vec{A}$ and $\vec{B}$ everywhere.

By equation 5.32 $\vec{A}(\vec{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{J(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$

Plugging in $\vec{J}(\vec{x}')$, $\vec{A}(\vec{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\sigma\omega a \sin\theta'\, \delta(r'-a)\,\hat{\phi}'\, r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'}{|\vec{x}-\vec{x}'|}$

Use the expansion $\dfrac{1}{|\vec{x}-\vec{x}'|} = 4\pi \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$

to get $\vec{A}(\vec{x}) = a\mu_0\sigma\omega \displaystyle\sum_l \sum_m \iiint \frac{r'^2 \sin^2\theta'\, \delta(r'-a)\,\hat{\phi}'\, dr'\, d\theta'\, d\phi'}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$

and plug in $\hat{\phi}' = -\sin\phi'\,\hat{x}' + \cos\phi'\,\hat{y}'$

So we can split this into the inside and outside sections:

$\rightarrow$ inside $r < r'$

$$\vec{A}_{in}(\vec{x}) = \sum_l \sum_m a^{2-l}\mu_0\sigma\omega \frac{r^l\, Y_{lm}(\theta,\phi)}{2l+1} \iint \sin^2\theta' (-\sin\phi'\,\hat{x}' + \cos\phi'\,\hat{y}')\, Y_{lm}^*(\theta',\phi')\, d\theta' d\phi'$$

$\rightarrow$ outside $r' < r$

$$\vec{A}_{out}(\vec{x}) = \sum_l \sum_m \frac{a^{3+l}}{r^{l+1}} \frac{\mu_0\sigma\omega\, Y_{lm}(\theta,\phi)}{(2l+1)} \iint \sin^2\theta' (-\sin\phi'\,\hat{x}' + \cos\phi'\,\hat{y}')\, Y_{lm}^*(\theta',\phi')\, d\theta' d\phi'$$

So, this comes down to solving the integral:

$$Int = Y_{lm}(\theta,\phi) \iint \sin\theta' (-\sin\theta' \sin\phi')\, Y_{lm}^*(\theta',\phi')\, d\theta' d\phi'\, \hat{x}$$
$$+ Y_{lm}(\theta,\phi) \iint \sin\theta' (\sin\theta' \cos\phi')\, Y_{lm}^*(\theta',\phi')\, d\theta' d\phi'\, \hat{y}$$

By 3.53, $Y_{lm}(\theta',\phi') = \sqrt{\dfrac{2l+1}{4\pi}\dfrac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta')\,(\cos(m\phi') + i\sin(m\phi'))$

Thus $Y_{11}(\theta',\phi') = \sqrt{\dfrac{3}{8\pi}}\,(-\sin\theta')(\cos\phi' + i\sin\phi')$

so $Re(Y_{11}(\theta',\phi')) = -\sqrt{\frac{3}{8\pi}}\sin\theta'\cos\phi'$ and $Im(Y_{11}(\theta',\phi')) = -\sqrt{\frac{3}{8\pi}}\sin\theta'\sin\phi'$

Then plug this into Int

$$Int = Im\left(Y_{lm}(\theta,\phi) \iint \sin\theta' \sqrt{\tfrac{8\pi}{3}}\, Y_{11}(\theta',\phi')\, Y_{lm}^*(\theta',\phi')\, d\theta' d\phi'\right)\hat{x}$$
$$- Re\left(Y_{lm}(\theta,\phi) \iint \sin\theta' \sqrt{\tfrac{8\pi}{3}}\, Y_{11}(\theta',\phi')\, Y_{lm}^*(\theta',\phi')\, d\theta' d\phi'\right)\hat{y}$$

Since $\iint Y_{11} Y^*_{\ell m} \, d\Omega = \delta_{\ell 1} \delta_{m,1}$,

$$Int = Im\left(Y_{\ell m}(\theta,\phi) \delta_{\ell,1} \delta_{m,1} \sqrt{\frac{8\pi}{3}} \right) \hat{x} - Re\left(Y_{\ell m}(\theta,\phi) \sqrt{\frac{8\pi}{3}} \delta_{\ell,1} \delta_{m,1} \right) \hat{y}$$

Now plug Int into $\vec{A}_{in}(\vec{x})$ and $\vec{A}_{out}(\vec{x})$

$$\vec{A}_{in}(\vec{x}) = \sqrt{\frac{8\pi}{3}} \frac{a\mu_0 \sigma \omega r}{3} \left(Im(Y_{11}(\theta,\phi)) \hat{x} - Re(Y_{11}(\theta,\phi)) \hat{y} \right)$$

$$= \cancel{\sqrt{\frac{8\pi}{3}}} \frac{a\mu_0 \sigma \omega r}{3} \left(-\cancel{\sqrt{\frac{3}{8\pi}}} \sin\theta \sin\phi \hat{x} + \cancel{\sqrt{\frac{3}{8\pi}}} \sin\theta \cos\phi \hat{y} \right)$$

$$= \frac{a\mu_0 \sigma \omega r}{3} \sin\theta \left(-\sin\phi \hat{x} + \cos\phi \hat{y} \right)$$

$$\boxed{\vec{A}_{in}(\vec{x}) = \tfrac{1}{3} a\mu_0 \sigma \omega r \sin\theta \, \hat{\phi}} \checkmark$$

Similarly, $\vec{A}_{out}(\vec{x}) = \cancel{\sqrt{\frac{8\pi}{3}}} \frac{a^4 \mu_0 \sigma \omega}{3r^2} \left(-\cancel{\sqrt{\frac{3}{8\pi}}} \sin\theta \sin\phi \hat{x} + \cancel{\sqrt{\frac{3}{8\pi}}} \sin\theta \cos\phi \hat{y} \right)$

$$\boxed{\vec{A}_{out}(\vec{x}) = \frac{a^4 \mu_0 \sigma \omega \sin\theta \, \hat{\phi}}{3r^2}} \checkmark$$

Use. $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$

$$\vec{B}_{in}(\vec{x}) = \vec{\nabla} \times \vec{A}_{in}(\vec{x}) = \frac{1}{r\sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin\theta \frac{a\mu_0 \omega r \sin\theta}{3} \right) \right] \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r a \mu_0 \omega r \sin\theta}{3} \right) \hat{\theta}$$

$$= \frac{2\cos\theta \, a\mu_0 \omega \hat{r}}{3} - \frac{2 a\mu_0 \omega \sin\theta \, \hat{\theta}}{3}$$

$$\boxed{\vec{B}_{in}(\vec{x}) = \frac{2}{3} a\mu_0 \omega \underbrace{(\cos\theta \hat{r} - \sin\theta \hat{\theta})}_{\hat{z}}} \checkmark$$

$$\vec{B}_{out}(\vec{x}) = \vec{\nabla} \times \vec{A}_{out}(\vec{x}) = \frac{1}{r\sin\theta} \left[\frac{\partial}{\partial\theta} \left(\frac{\sin\theta \, a^4 \mu_0 \sigma \omega \sin\theta}{3r^2} \right) \right] \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r a^4 \mu_0 \sigma \omega \sin\theta}{3r^2} \right) \hat{\theta}$$

$$= \frac{2\cos\theta}{3r^3} a^4 \mu_0 \sigma \omega \hat{r} + \frac{1}{3r^3} a^4 \mu_0 \sigma \omega \sin\theta \, \hat{\theta}$$

$$\boxed{\vec{B}_{out}(\vec{x}) = \frac{a^4 \mu_0 \sigma \omega}{3r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})} \checkmark$$

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