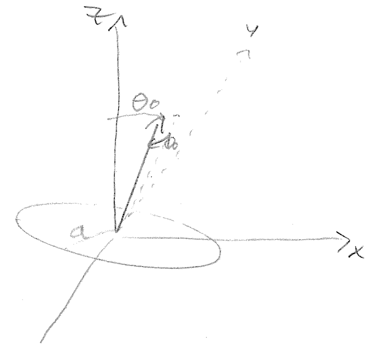
#2 Jackson 5.11

Given: $\vec{B}(\vec{x}) = B_0(1+\beta y)\hat{x} + B_0(1+\beta x)\hat{y}$

we must find $\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$



→ $J(\vec{x})$ is hard to find, but in a new (primed) coord. system where $\hat{z}'$ is normal to the loop, it's simple: $\vec{J}(\vec{x}') = \frac{I}{r'}\delta(r'-a)\,\delta(\theta'-\frac{\pi}{2})\,\hat{\phi}'$

→ Thus $\vec{F}(\vec{x}') = I\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{\delta(r'-a)}{r'}\delta(\theta'-\frac{\pi}{2})\hat{\phi}' \times \vec{B}(\vec{x}')\right) r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$

$= Ia\int_0^{2\pi} \hat{\phi}' \times \vec{B}(\vec{x}')\, d\hat{\phi}'$

Note $\hat{\phi}' = -\sin\phi'\,\hat{x}' + \cos\phi'\,\hat{y}'$

$\vec{B}(\vec{x}') = B_{x'}\hat{x}' + B_{y'}\hat{y}' + B_{z'}\hat{z}'$

$= Ia\int_0^{2\pi}(-\sin\phi'\hat{x}' + \cos\phi'\hat{y}') \times (B_{x'}\hat{x}' + B_{y'}\hat{y}' + B_{z'}\hat{z}')\, d\hat{\phi}'$

① $\vec{F}(\vec{x}') = Ia\int_0^{2\pi}\left[(\cos\phi' B_{z'})\hat{x} + (\sin\phi' B_{z'})\hat{y} - (\sin\phi' B_{y'} + \cos\phi' B_{x'})\right] d\phi'$

→ Now we need to find $B_{x'}$, $B_{y'}$, and $B_{z'}$ in terms of B. To do this I'll use rotation maticies.

Note $\vec{x} \to \vec{x}'$ means a rotation by θ_0 and ϕ_0

so we use $R_y(\theta_0)R_z(\phi_0) = \begin{bmatrix}\cos\theta_0 & 0 & \sin\theta_0\\ 0 & 1 & 0\\ -\sin\theta_0 & 0 & \cos\theta_0\end{bmatrix}\begin{bmatrix}\cos\phi_0 & \sin\phi_0 & 0\\ -\sin\phi_0 & \cos\phi_0 & 0\\ 0 & 0 & 1\end{bmatrix}$

$$\begin{bmatrix}B_{x'}\\ B_{y'}\\ B_{z'}\end{bmatrix} = \begin{bmatrix}\cos\theta_0\cos\phi_0 & \cos\theta_0\sin\phi_0 & -\sin\theta_0\\ -\sin\phi_0 & \cos\phi_0 & 0\\ \cos\phi_0\sin\theta_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0\end{bmatrix}\begin{bmatrix}B_x\\ B_y\\ B_z\end{bmatrix}$$

See Mathematica for this matrix multiplication

②
$B_{x'} = B_0(1+y\beta)\cos\phi_0\cos\theta_0 + B_0(1+x\beta)\cos\theta_0\sin\phi_0$

$B_{y'} = -B_0(1+y\beta)\sin\phi_0 + B_0(1+x\beta)\cos\phi_0$

$B_{z'} = -B_0(1+y\beta)\cos\phi_0\sin\theta_0 - B_0(1+x\beta)\sin\theta_0\sin\phi_0$

Yuck, but we still need x and y in terms of x' and y'.

as before

$$\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix} = \begin{bmatrix}\cos\theta_0\cos\phi_0 & \cos\theta_0\sin\phi_0 & -\sin\theta_0\\ -\sin\phi_0 & \cos\phi_0 & 0\\ \cos\phi_0\sin\theta_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

We need to turn this "inside-out" using the transpose: $\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}\cos\theta_0\cos\phi_0 & -\sin\phi_0 & \cos\phi_0\sin\theta_0\\ \cos\theta_0\sin\phi_0 & \cos\phi_0 & \sin\theta_0\sin\phi_0\\ -\sin\theta_0 & 0 & \cos\theta_0\end{bmatrix}\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix}$

we can just ignore the z-stuff since we only need x and y. Also, there is no z-dependance. Doing this multiplication in Mathematica, we get

(3) $$x = x'\cos\theta_0\cos\phi_0 - y'\sin\phi_0 \quad , \quad y = x'\cos\theta_0\sin\phi_0 + y'\cos\phi_0$$

Plug these into (2) on the previous page, and note $y' = a\sin\phi'$, $x' = a\cos\phi'$ (since we only integrate from $0 \to 2\pi$ in ϕ').

This yields $B_{x'}$, $B_{y'}$ and $B_{z'}$, which are output in Equation (4) of the attached Mathematica document. I won't transcribe them here b/c pencil lead is expensive..... ☺

→ Now it's time to get the Forces. Equation (1) tells us that

$$F_{x'} = Ia\int_0^{2\pi}\cos(\phi')B_{z'}d\phi' \quad , \quad F_{y'} = Ia\int_0^{2\pi}\sin(\phi')B_{z'}d\phi'$$

$$F_{z'} = -Ia\int_0^{2\pi}\left(\sin(\phi')B_{y'} + \cos(\phi')B_{x'}\right)d\phi'$$

All that's left to do is to plug in $B_{x'}$, $B_{y'}$ and $B_{z'}$, and evaluate the integrals. Mathematica is once again helpful. This gives us the following

(5) $$\begin{cases} F_{x'} = \frac{1}{2}a^2B_0I\pi\beta\sin(2\theta_0)\sin(2\phi_0) \\ F_{y'} = a^2B_0I\pi\beta\cos(2\phi_0)\sin(\theta_0) \\ F_{z'} = a^2B_0I\pi\beta\sin(2\phi_0)\sin^2(\theta_0) \end{cases}$$

→ This is great, but now we need to transform back to the unprimed coordinates. This is done through the transpose rotation matrix.

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\theta_0\cos\phi_0 & -\sin\phi_0 & \cos\phi_0\sin\theta_0 \\ \cos\theta_0\sin\phi_0 & \cos\phi_0 & \sin\theta_0\sin\phi_0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix} a^2B_0I\pi\beta\begin{bmatrix} \frac{1}{2}\sin(2\theta_0)\sin(2\phi_0) \\ \cos(2\phi_0)\sin(\theta_0) \\ \sin(2\phi_0)\sin^2(\theta_0) \end{bmatrix}$$

✓

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = a^2B_0I\pi\beta\begin{bmatrix} \sin\theta_0\sin\phi_0 \\ \sin\theta_0\cos\phi_0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\vec{F} = a^2B_0I\pi\beta\sin\theta_0(\sin\phi_0\hat{x} + \cos\phi_0\hat{y})}$$

Now, check against the approximation (5.69) $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$

per 5.57, for this plane loop, $|\vec{m}| = I\cdot(\text{Area}) = I\pi a^2$, and the direction, $\hat{a}$ is normal to the loop – i.e. $\hat{z}$ rotated by θ_0 and then ϕ_0. $\hat{a} = \sin\theta_0\sin\phi_0\hat{x} + \sin\theta_0\cos\phi_0\hat{y} + \cos\theta_0\hat{z}$.

Using $\vec{B} = B_0(1+\beta y)\hat{x} + B_0(1+\beta x)\hat{y}$ and $\vec{m} = I\pi a^2(\sin\theta_0\sin\phi_0\hat{x} + \sin\theta_0\cos\phi_0\hat{y} + \cos\theta_0\hat{z})$

we have $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = \vec{\nabla}\left(I\pi a^2\left(B_0(1+\beta y)\sin\theta_0\cos\phi_0 + B_0(1+\beta x)\sin\theta_0\sin\phi_0\right)\right)$

$\boxed{\vec{F} = I\pi a^2B_0\beta\sin\theta_0(\sin\phi_0\hat{x} + \cos\phi_0\hat{y})}$ ✓ Exactly the same as the exact technique.

Electrodynamics Homework 5 — Ben Levy

#2 Jackson Problem 5.11a

We begin by constructing the rotation matrix. inverse there-of I looked up how to do the rotations by angles ϕ_0 and θ_0. Note that throughout the notebook, I will use θ and ϕ to mean ϕ_0 and θ_0, while θp and ϕp will correspond to θ' and ϕ'. (p for prime). We get to the desired point by making a rotation of ϕ_0 around the ~~y~~ z axis (per my picture), and then a rotation of θ_0 about the ~~z~~ y-axis. The corresponding rotation matrices are rth and rphi ($R(\theta_0)$ and $R(\phi_0)$).

In[249]:=
```
rth = {{Cos[θ], 0, -Sin[θ]}, {0, 1, 0}, {Sin[θ], 0, Cos[θ]}};
```

In[250]:=
```
rphi = {{Cos[ϕ], Sin[ϕ], 0}, {-Sin[ϕ], Cos[ϕ], 0}, {0, 0, 1}};
rth // MatrixForm
rphi // MatrixForm
```

Out[251]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta] & 0 & -\mathrm{Sin}[\theta] \\ 0 & 1 & 0 \\ \mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] \end{pmatrix}$$

Out[252]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\phi] & \mathrm{Sin}[\phi] & 0 \\ -\mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The total rotation matrix then can be found using matrix multiplication. We get the following rotation matrix. It will prove extremely useful.

In[253]:=
```
rot = rth.rphi;
rot // MatrixForm
```

Out[254]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi] & \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi] & -\mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] & \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] & \mathrm{Cos}[\theta] \end{pmatrix}$$

Now that I've set up the machinery, My approach is to work entirely in my primed frame (as explained in the written portion), and then transform back at the end. Thus, I must transform the B-field to primed coordinates. The following is a matrix of the components of B in the unprimed frame:

In[255]:=
```
B = {B0 (1 + β y), B0 (1 + β x ), 0};
B // MatrixForm
```

Out[256]//MatrixForm=

$$\begin{pmatrix} \mathrm{B0}\ (1 + y\,\beta) \\ \mathrm{B0}\ (1 + x\,\beta) \\ 0 \end{pmatrix}$$

Now Bp is B' = Bx' + By' + Bz'. I got it by applying the rotation matrix. Here is a matrix of the elements.

The following is Equation 2

In[257]:=
```
Bp = rot.B;
Bp // MatrixForm
```

Out[258]//MatrixForm=

$$\begin{pmatrix} B0\ (1 + y\,\beta)\ Cos[\theta]\ Cos[\phi] + B0\ (1 + x\,\beta)\ Cos[\theta]\ Sin[\phi] \\ B0\ (1 + x\,\beta)\ Cos[\phi] - B0\ (1 + y\,\beta)\ Sin[\phi] \\ B0\ (1 + y\,\beta)\ Cos[\phi]\ Sin[\theta] + B0\ (1 + x\,\beta)\ Sin[\theta]\ Sin[\phi] \end{pmatrix}$$

Next we notice that these expressions still contain unprimed coordinates *x* and *y*. We can quickly fix this by writing *x* and *y* in terms of *x'* and *y'*. This is done by using the rotation matrix again. As explained in the written portion, we require the transpose of the rotation matrix since we are going from unprimed to primed this time. Note qp are the primed coordinates, and q are the unprimed coordinates.

The following yields Equation 3

In[259]:=
```
qp = {xp, yp, 0};
q = Transpose[rot].qp;
q // MatrixForm
```

Out[261]//MatrixForm=

$$\begin{pmatrix} xp\ Cos[\theta]\ Cos[\phi] - yp\ Sin[\phi] \\ yp\ Cos[\phi] + xp\ Cos[\theta]\ Sin[\phi] \\ -xp\ Sin[\theta] \end{pmatrix}$$

Noting that *y*' = *a* Sin[ϕ'] and x' = a Cos[ϕ'], we can now plug the coordinates from Equation 3 into Equation 2. Note that ϕp and θp are the primed angles (i.e. the source coordinates we need to integrate out). These are the complete expressions for the Magnetic field in the Primed coordinates.

The following yields Equation 4

In[262]:=
```
Bpx = (Bp[[1]] /. {x → q[[1]], y → q[[2]]} /. {xp → a * Cos[ϕp], yp → a * Sin[ϕp]}) //
  FullSimplify
Bpy = (Bp[[2]] /. {x → q[[1]], y → q[[2]]} /. {xp → a * Cos[ϕp], yp → a * Sin[ϕp]}) //
  FullSimplify
Bpz = (Bp[[3]] /. {x → q[[1]], y → q[[2]]} /. {xp → a * Cos[ϕp], yp → a * Sin[ϕp]}) //
  FullSimplify
```

Out[262]= $B0\ Cos[\theta]\ (Cos[\phi] + Sin[\phi] + a\,\beta\ (Cos[\theta]\ Cos[\phi p]\ Sin[2\,\phi] + Cos[2\,\phi]\ Sin[\phi p]))$

Out[263]= $B0\ (Cos[\phi] + a\,\beta\ Cos[\theta]\ Cos[2\,\phi]\ Cos[\phi p] - Sin[\phi]\ (1 + 2\,a\,\beta\ Cos[\phi]\ Sin[\phi p]))$

Out[264]= $B0\ Sin[\theta]\ (Cos[\phi] + Sin[\phi] + a\,\beta\ (Cos[\theta]\ Cos[\phi p]\ Sin[2\,\phi] + Cos[2\,\phi]\ Sin[\phi p]))$

From equation 1, we get the following expressions for the primed components of the force. Here we integrate out the primed coordinates.

The following yields Equation 5

In[265]:= $\text{Fxp} = \text{Ii} * \text{a} * \int_0^{2\pi} \text{Cos}[\phi p] * \text{Bpz} \, d\phi p$ // FullSimplify

$\text{Fyp} = \text{Ii} * \text{a} * \int_0^{2\pi} \text{Sin}[\phi p] * \text{Bpz} \, d\phi p$ // FullSimplify

$\text{Fzp} = -\text{Ii} * \text{a} * \int_0^{2\pi} (\text{Sin}[\phi p]\ \text{Bpy} + \text{Cos}[\phi p]\ \text{Bpx}) \, d\phi p$ // FullSimplify

Out[265]= $a^2\ \text{B0}\ \text{Ii}\ \pi\ \beta\ \text{Cos}[\theta]\ \text{Sin}[\theta]\ \text{Sin}[2\ \phi]$

Out[266]= $a^2\ \text{B0}\ \text{Ii}\ \pi\ \beta\ \text{Cos}[2\ \phi]\ \text{Sin}[\theta]$

Out[267]= $a^2\ \text{B0}\ \text{Ii}\ \pi\ \beta\ \text{Sin}[\theta]^2\ \text{Sin}[2\ \phi]$

The Final step is to translate the forces F_x', F_y' and F_z' out of the primed coordinate system. We can do this with the transpose rotation matrix. I will create a vector of primed forces (Fp), and then apply transpose(rot) to it to get F_x, F_y and F_z, arranged in a vector F.

In[268]:=
```
Fp = {Fxp, Fyp, Fzp};
F = Transpose[rot].Fp // MatrixForm // FullSimplify
```

Out[269]//MatrixForm=

$$\begin{pmatrix} a^2\ \text{B0}\ \text{Ii}\ \pi\ \beta\ \text{Sin}[\theta]\ \text{Sin}[\phi] \\ a^2\ \text{B0}\ \text{Ii}\ \pi\ \beta\ \text{Cos}[\phi]\ \text{Sin}[\theta] \\ 0 \end{pmatrix}$$

(b) $\vec{N} = \vec{m} \times \vec{B}(0)$ is the equation for Torque to lowest order

$\vec{B}(0) = B_0 \hat{x} + B_0 \hat{y}$, and $\vec{m} = I \cdot \pi a^2 (\sin\theta_0 \sin\phi_0\, \hat{x} + \sin\theta_0 \cos\phi_0\, \hat{y} + \cos\theta_0\, \hat{z})$

$$\boxed{\vec{N} = B_0 I \pi a^2 \left(\cos\theta\, \hat{x} - \cos\theta\, \hat{y} + (\cos\phi \sin\theta - \sin\phi \sin\theta)\, \hat{z} \right)}$$ ✓

In this case, $\vec{B}$ is linear in the coordinate terms (x and y), so any higher derivatives should just leave zero. This means that the additional terms in the $\vec{B}$ expansion do not matter, so higher order terms cancel out.

✓

For other shapes it is unlikely they would vanish due to the lack of symmetry.

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