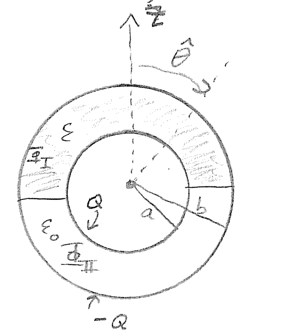
#4 Jackson 4.10

(a) We are given the situation at the right where we have an azimuthaly symmic charge distro with Thick half-spherical shells.



We call The potential in the region w/ ε, Φ_I, and the region with ε_0 Φ_{II}. We will proceed by guessing the form of the potential, then applying B.C.'s to get constats.

Start with 3.33 (since m=0 w/ azimuthal symmetry, and we are "on-axis")

$$\Phi(r,\theta) = \frac{1}{4\pi\varepsilon}\sum_{l=0}^{\infty}\left(Ar^l - Br^{-(l+1)}\right)P_l(\cos\theta).$$

Thus $$\Phi_I = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(Ar^l - Br^{-(l+1)}\right)P_l(\cos\theta)$$

and $$\Phi_{II} = \frac{1}{4\pi\varepsilon}\sum_{l=0}^{\infty}\left(Cr^l - Dr^{-(l+1)}\right)P_l(\cos\theta)$$

To eliminate these unknown constats (A, B, C, D), use Boundary Conditions:

→ Constat potential on inner and outer spherical shell (b/c they are conductors)

$\Phi(a,\theta) = V_a = V_a P_0(\cos\theta)$ and $\Phi(b,\theta) = V_b = V_b P_0(\cos\theta)$

Thus $\Phi(r,\theta) = \frac{1}{4\pi\varepsilon}\sum_{l=0}^{\infty}\left(Ar^l - Br^{-(l+1)}\right)P_l(\cos\theta) = V_a P_0(\cos\theta)$ (V_a or b)

and so, by orthogonality, only $l=0$.

so $\Phi_I = \frac{1}{4\pi\varepsilon_0}\left(A - \frac{B}{r}\right)$ and $\Phi_{II} = \frac{1}{4\pi\varepsilon}\left(C - \frac{D}{r}\right)$

⇒ Potential must be constat across $\theta = \frac{\pi}{2}$ interface.

$\Phi_I = \Phi_{II}\big|_{\theta=\frac{\pi}{2}}$ (satisfied trivially...) $\frac{A}{\varepsilon_0} - \frac{B}{r\varepsilon_0} = \frac{C}{\varepsilon} - \frac{D}{r\varepsilon}$

Equate coefficients to r^0 and r^{-1} terms ⇒ $\frac{A}{\varepsilon_0} = \frac{C}{\varepsilon}$ and $\frac{B}{\varepsilon_0} = \frac{D}{\varepsilon}$.

And plug in the simplified coeficients:

$$V_a = \frac{1}{4\pi\varepsilon_0}\left(C\frac{\varepsilon_0}{\varepsilon} - \frac{B}{a}\right) = \frac{1}{4\pi\varepsilon}\left(C - \frac{B}{a}\frac{\varepsilon}{\varepsilon_0}\right)$$

$$V_b = \frac{1}{4\pi\varepsilon_0}\left(C\frac{\varepsilon_0}{\varepsilon} - \frac{B}{b}\right) = \frac{1}{4\pi\varepsilon}\left(C - \frac{B}{b}\frac{\varepsilon}{\varepsilon_0}\right)$$

so $V_b - V_a = \frac{B}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$ either way. ⇒ $B = \frac{4\pi\varepsilon_0(V_b - V_a)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

and since $D = \frac{\varepsilon}{\varepsilon_0}B$, $D = \frac{4\pi\varepsilon(V_b - V_a)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

Now, our potential is fully figured out. We're looking for E-field though: $\vec{E} = -\vec{\nabla}\Phi$

$\vec{E}_I = -\vec{\nabla}\Phi_I = \frac{B}{4\pi\varepsilon_0 r^2}\hat{r}$, and $\vec{E}_{II} = -\vec{\nabla}\Phi_{II} = \frac{D}{4\pi\varepsilon r^2}\hat{r}$ ⇒ $\vec{E}_I = \vec{E}_{II} = \frac{(V_b - V_a)}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2}\hat{r}$

But we still have those pesky, unknown V_a and V_b.

We will rewrite this in terms of charge using Gauss' Law:

$$Q_{enc} = \int \vec{D}\cdot d\vec{a} = \varepsilon\int \vec{E}\cdot d\vec{a}$$

→

so in our case, we will integrate only over the inner surface where $Q_{enc} = Q$.

$$Q = \int_{I,\, r=a} \frac{\varepsilon_0 (V_b - V_a)}{\left(\frac{1}{a} - \frac{1}{b}\right) r^2} \hat{r} \cdot d\vec{a} + \int_{II,\, r=a} \frac{\varepsilon (V_b - V_a)}{\left(\frac{1}{a} - \frac{1}{b}\right) r^2} \hat{r} \cdot d\vec{a}$$

$$Q = \frac{V_b - V_a}{\left(\frac{1}{a} - \frac{1}{b}\right)} \left[\int_0^{2\pi} \int_{\pi/2}^{\pi} \frac{\varepsilon_0}{r^2} \sin\theta \, d\theta \, d\phi + \int_0^{2\pi} \int_0^{\pi/2} \frac{\varepsilon}{r^2} \sin\theta \, d\theta \, d\phi \right] = \frac{(V_b - V_a) 2\pi}{a^2 \left(\frac{1}{a} - \frac{1}{b}\right)} (\varepsilon_0 + \varepsilon)$$

rearrange $\dfrac{Q\left(\frac{1}{a} - \frac{1}{b}\right)}{2\pi a^2 \left(\frac{\varepsilon}{a^2} + \frac{\varepsilon}{a^2}\right)} = V_a - V_b$

Thus $\boxed{\vec{E} = \dfrac{Q}{2\pi(\varepsilon_0 + \varepsilon) r^2} \hat{r}}$ ✓ Everywhere between spheres! Crazy. Must have to do w/ the fact that the conductors allow charge to rearrange to cancel everything out...

(b) To find σ on the inner conductor, use $(D_2 - D_1) \cdot \hat{n}_{21} = \sigma$

Inside the solid, conducting sphere, $\vec{D} = 0$, of course, so $2 \to$ in, $1 \to$ out

$-\varepsilon \vec{E}(a) \cdot (-\hat{r}) = \sigma_I$ and $-\varepsilon_0 \vec{E}(a) \cdot (-\hat{r}) = \sigma_{II}$ (it will be differ in the two regions, evidently). ✓ ✓

$$\boxed{\sigma_I = \frac{Q \varepsilon}{2\pi(\varepsilon_0 + \varepsilon) a^2}, \quad \sigma_{II} = \frac{Q \varepsilon_0}{2\pi(\varepsilon_0 + \varepsilon) a^2}}$$

(c) To get σ_{pol}, we use 4.46: $\sigma_{pol} = -(P_2 - P_1) \cdot \hat{n}_{21}$

Now $P_i = (\varepsilon_i - \varepsilon_0) E_i \Rightarrow P_{out} = (\varepsilon - \varepsilon_0) E_{out}\big|_{r=a}$ $P_{in} = (\varepsilon_0 - \varepsilon_0) E_{in} = 0$. (in the center)

Thus $\sigma_{pol}\big|_{r=a} = -(\varepsilon - \varepsilon_0) \vec{E} \cdot (-\hat{r}) = (\varepsilon - \varepsilon_0)|E| = \boxed{\dfrac{Q(\varepsilon - \varepsilon_0)}{2\pi(\varepsilon_0 + \varepsilon) a^2}}$ ✓

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