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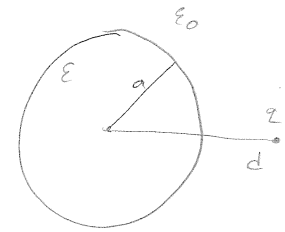
Jackson 4.9 (a)



Outside: Use images. Put q' at $y = a^2/d$ as on hwk 2,

So $$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{x} - d\hat{y}|} + \frac{q'}{|\vec{x} - (\frac{a^2}{d})\hat{y}|}\right)$$

and use the fact that we have spherically symmetric B.C.'s to

get $$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

Thus $$\Phi_{out}^{>}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\sum_{l=0}^{\infty} \frac{q d^l}{x^{l+1}} P_l(\cos\theta) + \frac{q'(\frac{a^2}{d})^l}{x^{l+1}} P_l(\cos\theta)\right) \quad |\vec{x}| > d$$

$$\longrightarrow \boxed{\Phi_{out}^{>}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{r^{l+1}}\left(q d^l + q'\frac{a^{2l}}{d^l}\right)\right) \quad |\vec{x}| > d} \checkmark$$

$$\longrightarrow \text{and } \boxed{\Phi_{out}^{<}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{d^{l+1}}\left(\frac{q r^l}{1} + \frac{q' a^{2l} d}{r^{l+1}}\right)\right) \quad |\vec{x}| < d}$$

Inside The effects are not quite that of a pt. charge q at d, but there is no reason to move the pt. charge. Place q'' at d instead.

$$\longrightarrow \text{Then } \boxed{\Phi_{in} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q''}{|\vec{x} - d\hat{y}|}\right) = \frac{q''}{4\pi\varepsilon}\sum_{l=0}^{\infty} \frac{r^l}{d^{l+1}} P_l(\cos\theta)} \checkmark$$

Now all we have to do is find q' and q'' in terms of q

Boundary conditions to the rescue

① On surface: $(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = \sigma$

② Tangential components are equal: $\vec{E}_1\cdot\hat{\theta} = \vec{E}_2\cdot\hat{\theta}$

① $\vec{D} = \epsilon\vec{E}$, set $2\to$ out, $1\to$ in, and $\sigma = 0$ b/c no charge on surface.

$$(\varepsilon_0\vec{E}_{out} - \varepsilon\vec{E}_{in})\cdot\hat{r} = 0 \Rightarrow \varepsilon_0\vec{E}_{out}\cdot\hat{r} = \varepsilon\vec{E}_{in}\cdot\hat{r} \Rightarrow \left.\frac{\varepsilon_0\,\partial\Phi_{out}^{<}}{\partial r}\right|_{r=a} = \left.\frac{\varepsilon\,\partial\Phi_{in}}{\partial r}\right|_{r=a}$$

We can Do this term by term, since summations are the same on both sides.

$$\varepsilon\left(\frac{q''}{4\pi\varepsilon}\, l\,\frac{r^{l-1}}{d^{l+1}} P_l(\cos\theta)\right) = \varepsilon_0\left(\frac{q}{4\pi\varepsilon_0}\, l\,\frac{r^{l-1}}{d^{l+1}} P_l(\cos\theta) - \frac{q'}{4\pi\varepsilon_0}(l+1)\frac{r^{-l-2} a^{2l}}{d^l} P_l(\cos\theta)\right)\Bigg|_{r=a}$$

$$\varepsilon\frac{q''}{\varepsilon} l r^{l-1} = \left(\frac{q}{\varepsilon_0} l r^{l-1} - \frac{q'}{\varepsilon_0}\frac{d a^{2l}}{r^{l+2}} - \frac{q'}{\varepsilon_0} l \frac{d a^{2l}}{r^{l+2}}\right)\varepsilon_0\Bigg|_{r=a}$$

$$q'' r^{l-1} = q r^{l-1} - q'\frac{d a^{2l}}{l r^{l+2}} - q'\frac{d a^{2l}}{r^{l+2}}\Bigg|_{r=a} \Rightarrow q'' a^{l-1} = q a^{l-1} - q'\frac{d}{l}a^{l-2} - q' d a^{l-2}$$

$$\Rightarrow q'' = q - q'\frac{d}{la} - q'\frac{d}{a} = q - q'\left(\frac{d}{la} + \frac{d}{a}\right)$$

② $$\left.\frac{\partial\Phi_{in}}{\partial\theta}\right|_{r=a} = \left.\frac{\partial\Phi_{out}^{<}}{\partial\theta}\right|_{r=a} \Rightarrow \text{(term by term)} \quad \frac{q''}{\varepsilon}\frac{a^l}{d^{l+1}} = \frac{1}{\varepsilon_0 d^{l+1}}\left(q a^l + q' a^{l-1} d\right)$$

$$\frac{q''}{\varepsilon} = \frac{1}{\varepsilon_0}\left(q + \frac{q' d}{a}\right)$$

Solve these two equations simultaneously for q' and q'' (see Mathematica).

$$\boxed{q' = \frac{-a l q(\varepsilon - \varepsilon_0)}{d(l\varepsilon + \varepsilon_0 + l\varepsilon_0)}}, \quad \boxed{q'' = \frac{(1+2l)q\varepsilon}{l\varepsilon + \varepsilon_0 + l\varepsilon_0}} \checkmark$$

(b) Near the center of the sphere, r is small, so r<<d, and $\frac{r}{d} << 1$

so write $\Phi_{in} = \frac{q''}{4\pi\varepsilon d}\sum_{\ell=0}^{\infty}\left(\frac{r}{d}\right)^{\ell} P_{\ell}(\cos\theta) \overset{\text{expand}}{\approx} \frac{q''}{4\pi\varepsilon d}\left[1+\frac{r}{d}(\cos\theta)\right]$

$$= \frac{(1+2\ell)q}{\ell\varepsilon+\varepsilon_0+\ell\varepsilon_0}\frac{1}{d4\pi}\left[1+\frac{r}{d}(\cos\theta)\right]$$

and note $r\cos\theta = y$ (z in standard spherical coordinates, but you're at liberty to do as you please!)

$$= \frac{(1+2\ell)q}{\ell\varepsilon+\varepsilon_0+\ell\varepsilon_0}\frac{1}{4\pi d}\left[1+\frac{y}{d}\right]$$

and $\vec{E}_{in} = -\frac{\partial}{\partial y}\Phi_{in}\hat{y} = \frac{-(1+2\ell)q}{4\pi d^2(\ell\varepsilon+\varepsilon_0+\ell\varepsilon_0)}\hat{y}$, $\ell = 1$, $= \boxed{\frac{-q}{4\pi\varepsilon_0 d^2}\left[\frac{3}{\frac{\varepsilon}{\varepsilon_0}+2}\right]\hat{y}}$ ✓

(c) Now deal with the limit $\varepsilon/\varepsilon_0 \to \infty$

$$\Phi_{in} = \frac{q}{4\pi\varepsilon_0 d}\left(\frac{1+2\ell}{\ell\frac{\varepsilon}{\varepsilon_0}+1+\ell}\right)\sum_{\ell=0}^{\infty}\left(\frac{r^{\ell}}{d^{\ell}}\right)P_{\ell}(\cos\theta)$$

all terms go to zero except $\ell=0$. Nice!

$$\Phi_{in\,\varepsilon/\varepsilon_0\to\infty} = \frac{q}{4\pi\varepsilon_0 d}\left(\frac{1}{1}\right)\left(\frac{r}{d}\right)^{0} = \boxed{\frac{q}{4\pi\varepsilon_0 d}}$$ ✓

and since $\vec{E}_{in\,\varepsilon/\varepsilon_0\to\infty} = -\vec{\nabla}\Phi_{in\,\varepsilon/\varepsilon_0\to\infty} = \boxed{0}$, we get the same result as that for a conducting sphere. It shields the inside from the electric field, and maintains constant potential due completely to external charge.

for Φ_{out}, $q'_{\varepsilon/\varepsilon_0\to 0} = \frac{1}{d\ell}q\ell a = \frac{qa}{d}$

Thus $\Phi_{out}^{>}(r,\theta)_{\varepsilon/\varepsilon_0\to\infty} = \frac{q}{4\pi\varepsilon_0}\left(\sum_{\ell=0}^{\infty}\frac{P_{\ell}(\cos\theta)}{r^{\ell+1}}\left(d^{\ell}+\frac{a^{2\ell+1}}{d^{\ell+1}}\right)\right)$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{a/d}{r}+\frac{1}{|\mathbf{r}-d\hat{y}|}-\frac{a/d}{|\mathbf{r}-\frac{a^2}{d}\hat{y}|}\right)$$ ✓

Just as we have done 1 Million times with normal image charges.

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Mathematica for part a

I use q'' = q2 and q' = q1

In[109]:= $\text{Solve}\left[\left\{q2 == q - q1\left(\frac{d}{L*a}+\frac{d}{a}\right), \frac{q2}{\varepsilon} == \frac{1}{\varepsilon 0}\left(q+\frac{q1*d}{a}\right)\right\}, \{q1, q2\}\right]$

Out[109]= $\left\{\left\{q1 \to -\frac{a\,L\,q\,(\varepsilon-\varepsilon 0)}{d\,(L\,\varepsilon+\varepsilon 0+L\,\varepsilon 0)}, q2 \to \frac{(1+2L)\,q\,\varepsilon}{L\,\varepsilon+\varepsilon 0+L\,\varepsilon 0}\right\}\right\}$