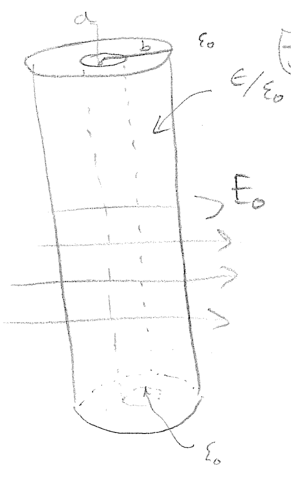


Jackson 4.8 a

We begin w/ the general cylindrical solution for when there is no z-dependence, and ϕ ranges all the way from 0 to 2π.

$$\Phi(\rho,\phi)=\sum_{m=0}^{\infty}(a_m\rho^{m}+b_m\rho^{-m})(A_m\cos(m\phi)+B_m\sin(m\phi))$$

Φ inside. Φ must be finite as $\rho\to0$, so $b_m=0$ and $b_0=0$.

$$\Phi_{in}(\rho,\phi)=\sum_{m=0}^{\infty}a_m\rho^{m}(A_m\cos(m\phi)+B_m\sin(m\phi)$$

Combined constants (redefine these vars - we'll make new ones in subsequent cases)

$$\longrightarrow \Phi_{in}(\rho,\phi)=\sum_{m=0}^{\infty}\rho^{m}(A_m\cos(m\phi)+B_m\sin(m\phi))$$

Φ outside

Φ must not blow up, so $a_m=0$ for $m>0$

$$\Phi_{out}(\rho,\phi)=a_0+\sum_{m=0}^{\infty}(b_m\rho^{-m})(A_m\cos(m\phi)+B_m(\sin(m\phi)))=a_0+\sum_{m=0}^{\infty}\rho^{-m}(C_m\cos(m\phi)+D_m\sin(m\phi))$$

as $\rho\to\infty$, $\Phi_{out}\to a_0$. All this potential is due to the E-field ($E_0\cos\phi$).

The potential then is $a_0=-E_0\rho\cos\phi$

$$\longrightarrow \Phi_{out}(\rho,\phi)=-E_0\rho\cos\phi+\sum_{m=0}^{\infty}\rho^{-m}(C_m\cos(m\phi)+D_m\sin(m\phi)$$

Φ middle ($a<\rho<b$) No new info: $\longrightarrow \Phi_{mid}(\rho,\phi)=\sum_{m=0}^{\infty}\rho^{m}(E_m\cos m\phi+F_m\sin m\phi)+\rho^{-m}(G_m C(m\phi)+H_m S(m\phi))$

B.C.'s $\quad \epsilon_+\vec{E}_+\cdot\hat{n}_+=\epsilon_-\vec{E}_-\cdot\hat{n}_-$ ← Normal $\quad$ Tangential → $E_+=E_-$

Inside Boundary ① $\quad \epsilon_0\dfrac{\partial\Phi_{in}}{\partial\rho}\Big|_{\rho=a}=\epsilon\dfrac{\partial\Phi_{mid}}{\partial\rho}\Big|_{\rho=a}\Rightarrow\dfrac{\epsilon_0}{\epsilon}\sum_{m=0}m a^{m-1}(A_m C(m\phi)+B_m S(m\phi))=\sum_{m=0}m a^{m-1}(E_m C(m\phi)+F_m S(m\phi))$

$$-ma^{-m-1}(G_m C(m\phi)+H_m S(m\phi))$$

$$0=\sum_{m=0}^{\infty}\cos(m\phi)\left(\tfrac{\epsilon_0}{\epsilon}ma^{m-1}A_m-ma^{m-1}E_m+ma^{-m-1}G_m\right)+\sin(m\phi)\left(ma^{m-1}B_m\tfrac{\epsilon_0}{\epsilon}-ma^{m-1}F_m+ma^{-m-1}H_m\right)$$

$$0=\sum_{m=0}^{\infty}\cos(m\phi)\,ma^{m-1}\left(\tfrac{\epsilon_0}{\epsilon}A_m-E_m+a^{-2m}G_m\right)+\sin(m\phi)\,ma^{m-1}\left(\tfrac{\epsilon_0}{\epsilon}B_m-F_m+a^{-2m}H_m\right)$$

$\cos(m\phi)\neq\sin(m\phi)\neq0\ \forall m$, so $\tfrac{\epsilon_0}{\epsilon}A_m-E_m+a^{-2m}G_m=0$ and $\tfrac{\epsilon_0}{\epsilon}B_m-F_m+a^{-2m}H_m=0$

② $\quad \dfrac{\partial\Phi_{in}}{\partial\phi}\Big|_{\rho=a}=\dfrac{\partial\Phi_{out}}{\partial\phi}\Big|_{\rho=a}\Rightarrow\sum_{m=0}^{\infty}a^{m}(-A_m m\sin(m\phi)+B_m m\cos(m\phi))$

$$=\sum_{m=0}^{\infty}a^{m}m(F_m\cos(m\phi)-E_m\sin(m\phi))+a^{-m}m(H_m\cos(m\phi)-G_m\sin(m\phi))$$

$$0=\sum_{m=0}^{\infty}a^{m}m\left(\cos(m\phi)(B_m-F_m-H_m a^{-2m})-\sin(m\phi)(A_m-E_m-G_m a^{-2m})\right)$$

by same logic, $A_m-E_m-a^{-2m}G_m=0$ and $B_m-F_m-a^{-2m}H_m=0$

Outside Boundary ① $\left.\frac{\varepsilon \, \partial \Phi_{mid}}{\partial \rho}\right|_{\rho=b} = \varepsilon_0 \left.\frac{\partial \Phi_{out}}{\partial \rho}\right|_{\rho=b}$

$$\Rightarrow \sum_{m=0}^{\infty}\Big[m b^{m-1}(E_m \cos m\phi + F_m \sin m\phi) - m b^{-m-1}(G_m \cos m\phi + H_m \sin m\phi)$$

$$+ \frac{\varepsilon_0}{\varepsilon}\left(m b^{-m-1}(C_m \cos m\phi + D_m \sin m\phi)\right)\Big] + \frac{\varepsilon_0}{\varepsilon} E_0 \cos\phi$$

$$\Rightarrow \sum_{m=0}^{\infty} m b^{-m-1}\left(\cos m\phi\left(b^{2m}E_m - G_m + \frac{\varepsilon_0}{\varepsilon}C_m\right) + \sin m\phi\left(b^{2m}F_m - H_m + \frac{\varepsilon_0}{\varepsilon}D_m\right)\right) + E_0\cos\phi$$

(so $b^{2m}E_m - G_m + \frac{\varepsilon_0}{\varepsilon}C_m = -\frac{\varepsilon_0}{\varepsilon}b^2 E_0 \delta_{m,1}$ and $b^{2m}F_m - H_m + \frac{\varepsilon_0}{\varepsilon}D_m = 0$)

② $\left.\frac{\partial \Phi_{mid}}{\partial \phi}\right|_{\rho=b} = \left.\frac{\partial \Phi_{out}}{\partial \phi}\right|_{\rho=b} \Rightarrow \sum_{m=0}^{\infty} b^m m(F_m \cos(m\phi) - E_m \sin m\phi) + b^{-m} m (H_m \cos m\phi - G_m \sin m\phi)$

$$= \sum_{m=0}^{\infty}\left[b^{-m} m \left(D_m \cos(m\phi) - C_m \sin(m\phi)\right)\right] + E_0 \sin\phi \, b$$

$$0 = \sum_{m=0}^{\infty}\left[b^{-m} m\left(\cos(m\phi)\left(b^{2m}F_m - H_m - D_m\right) - \sin m\phi\left(b^{2m}E_m - G_m - C_m\right)\right)\right] + E_0 \sin\phi \, b$$

$$b^{2m}F_m - H_m - D_m = 0 \qquad b^{2m}E_m - G_m - C_m = -b^2 E_0 \delta_{m,1}$$

→ The B_m, F_m, H_m, D_m system is only satisfied when $0 = B_m = F_m = H_m = D_m \ \forall m$ (see mathematica).

→ The A_m, E_m, G_m, C_m system is only satisfied when $0 = A_m = E_m = G_m = C_m \ \forall m$ (see Mathematica) EXCEPT if $m=1$. Then we get

$$A_1 = \frac{4b^2 E_0 \varepsilon \varepsilon_0}{(\varepsilon+\varepsilon_0)(a^2(\varepsilon-\varepsilon_0) - b^2(\varepsilon+\varepsilon_0))}, \quad E_1 = \frac{-2b^2 E_0 \varepsilon_0}{a^2(\varepsilon_0 - \varepsilon) + b^2(\varepsilon+\varepsilon_0)}$$

$$G_1 = \frac{2a^2 b^2 E_0(\varepsilon-\varepsilon_0)\varepsilon_0}{(\varepsilon+\varepsilon_0)(a^2(\varepsilon-\varepsilon_0) - b^2(\varepsilon+\varepsilon_0))}, \quad C_1 = \frac{b^2 E_0(\varepsilon-\varepsilon_0)}{\varepsilon+\varepsilon_0}$$

✓ $\Phi_{\rho<a}(\rho,\phi) = \rho A_1 \cos(\phi)$ ✓ $\Phi_{\rho>b}(\rho,\phi) = -E_0 \rho\cos\phi + \frac{1}{\rho}C_1\cos(\phi)$

✓ $\Phi_{a<\rho<b}(\rho,\phi) = \rho E_1 \cos(\phi) + \frac{1}{\rho}G_1\cos(\phi)$

Electric field : $E = -\nabla\Phi = -\left(\frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}\right)\Phi$

$\vec{E}_{\rho<a}(\rho,\phi) = -A_1\cos(\phi)\hat{\rho} + A_1\sin(\phi)\hat{\phi} = \boxed{A_1\hat{x}}$ → b/c $\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi} = \hat{x}$

(all seem reasonable)

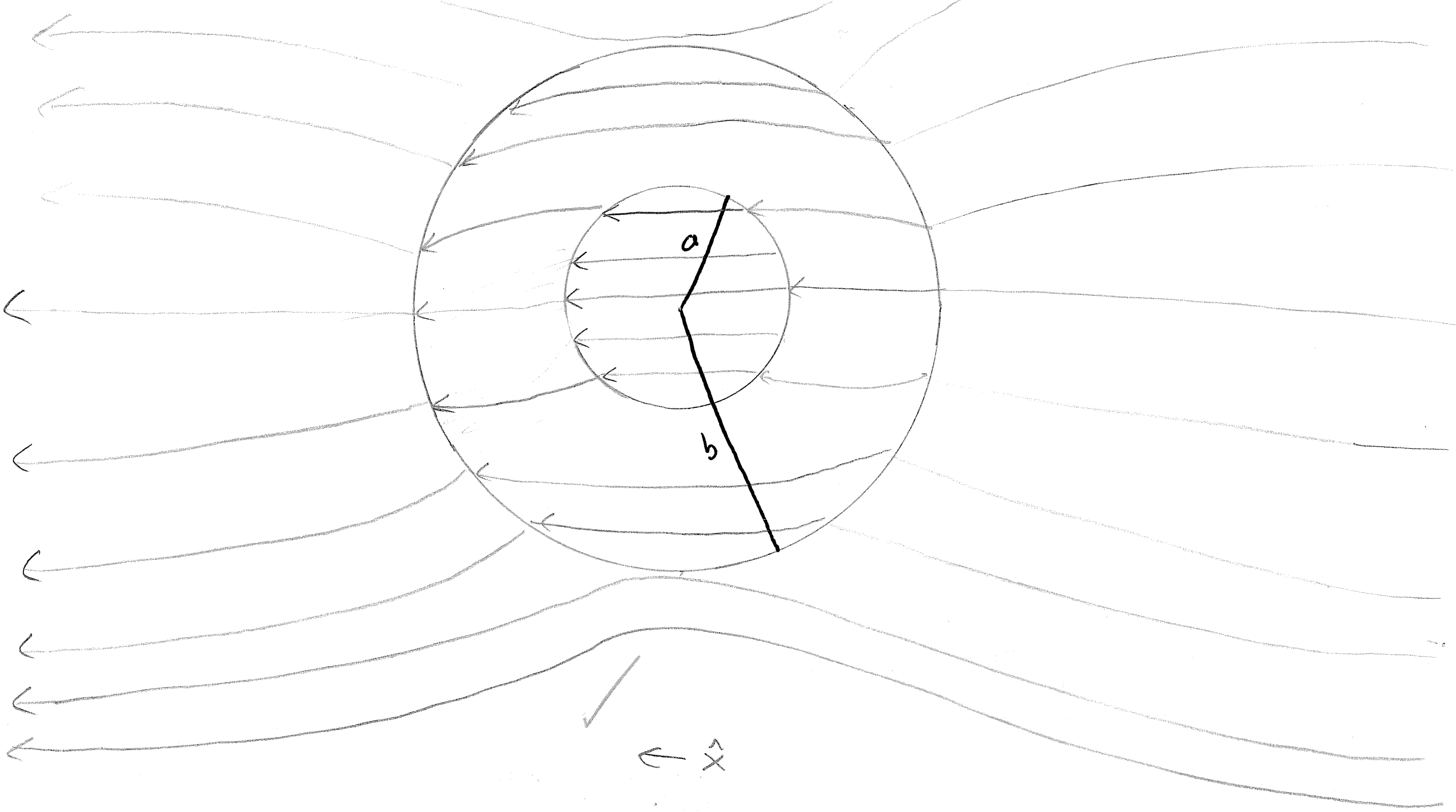
$$\vec{E}_{a<\rho<b}(\rho,\phi) = -\left[\left(E_1\cos(\phi) - \rho^{-2}G_1\cos(\phi)\right)\hat{\rho} + \left(-E_1\sin(\phi) - \frac{1}{\rho^2}G_1\sin(\phi)\right)\hat{\phi}\right]$$

$$= \left(\rho^{-2}G_1 - E_1\right)\cos(\phi)\hat{\rho} + \left(\rho^{-2}G_1 + E_1\right)\sin(\phi)\hat{\phi}$$

$$\vec{E}_{\rho>b}(\rho,\phi) = -\left[\left(E_0\cos\phi - \rho^{-2}C_1\cos\phi\right)\hat{\rho} + \left(E_0\sin\phi + \frac{1}{\rho^2}C_1\sin(\phi)\right)\hat{\phi}\right]$$

$$= \left(\rho^{-2}C_1 - E_0\right)\cos(\phi)\hat{\rho} + \left(\rho^{-2}C_1 - E_0\right)\sin(\phi)\hat{\phi}$$

(b) The "lines of Force" are just the contours of $\vec{E}$. Here $b \simeq 2a$:



(c) Limiting Case: Solid dielectric cylinder   Then $a \to 0$

$A_1 = \dfrac{-4E_0\varepsilon\varepsilon_0}{(\varepsilon+\varepsilon_0)^2}$ $E_1 = \dfrac{-2E_0\varepsilon_0}{(\varepsilon+\varepsilon_0)}$, $G_1 = 0$, $C_1 = \dfrac{b^2E_0(\varepsilon-\varepsilon_0)}{(\varepsilon+\varepsilon_0)}$

so $\vec{E}_{\rho<b}(\rho,\phi) = -E_1\cos(\phi)\hat{\rho} + E_1\sin(\phi)\hat{\phi} = \dfrac{+2\varepsilon_0}{(\varepsilon+\varepsilon_0)}E_0\hat{x}$ ✓

$$\vec{E}_{\rho>b}(\rho,\phi) = \left(\frac{1}{\rho^2}\frac{b^2E_0(\varepsilon-\varepsilon_0)}{\varepsilon+\varepsilon_0} - E_0\right)\cos(\phi)\hat{\rho} + \rho^{-2}\left(\frac{b^2E_0(\varepsilon-\varepsilon_0)}{(\varepsilon+\varepsilon_0)} - E_0\right)\sin\phi\,\hat{\phi}$$ ✓

SO, the field becomes constant in the solid cylinder.

Limiting Case: Cylinder Cavity   Then $b \to \infty$

$\vec{E}_{\rho<a} = \dfrac{4E_0\varepsilon\varepsilon_0}{(\varepsilon+\varepsilon_0)^2}\hat{x}$ ← still constant ✓

$\vec{E}_{\rho>a} = \dfrac{2\varepsilon_0}{(\varepsilon+\varepsilon_0)}E_0\hat{x} + \rho^{-2}\left(\dfrac{2\varepsilon_0(\varepsilon-\varepsilon_0)}{(\varepsilon+\varepsilon_0)}\right)E_0\left(\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi}\right)$ ✓

(10)

Solveing for the B,F,H,D system of equations

In[82]:= $\text{Solve}\left[\left\{\frac{\varepsilon 0}{\varepsilon} B - F + a^{-2} H == 0,\ B - F - a^{-2} H == 0,\ b^2 F - H + \frac{\varepsilon 0}{\varepsilon} dD == 0,\ b^2 F - H - dD == 0\right\}, \{B, F, H, dD\}\right]$

Out[82]= $\{\{B \to 0,\ F \to 0,\ H \to 0,\ dD \to 0\}\}$

Solveing for the A,E,G,C system when m ≠ 1.

In[81]:= $\text{Solve}\left[\left\{\frac{\varepsilon 0}{\varepsilon} A - eE + a^{-2} G == 0,\ A - eE - a^{-2} G == 0,\ b^2\, eE - G + \frac{\varepsilon 0}{\varepsilon} C == 0,\ b^2\, eE - G - C == 0\right\}, \{A, eE, G, C\}\right]$

Out[81]= $\{\{A \to 0,\ eE \to 0,\ G \to 0,\ C \to 0\}\}$

But when m = 1, we have

In[80]:= $\text{Solve}\left[\left\{\frac{\varepsilon 0}{\varepsilon} A - eE + a^{-2} G == 0,\ A - eE - a^{-2} G == 0,\ b^2\, eE - G + \frac{\varepsilon 0}{\varepsilon} C == -\frac{\varepsilon 0}{\varepsilon} b^2\, E0,\ b^2\, eE - G - C == -b^2\, E0\right\}, \{A, eE, G, C\}\right]$ // FullSimplify

Out[80]= $\left\{\left\{A \to \frac{4\, b^2\, E0\, \varepsilon\, \varepsilon 0}{(\varepsilon + \varepsilon 0)\left(a^2 (\varepsilon - \varepsilon 0) - b^2 (\varepsilon + \varepsilon 0)\right)},\ eE \to -\frac{2\, b^2\, E0\, \varepsilon 0}{a^2 (-\varepsilon + \varepsilon 0) + b^2 (\varepsilon + \varepsilon 0)},\ G \to \frac{2\, a^2\, b^2\, E0\, (\varepsilon - \varepsilon 0)\, \varepsilon 0}{(\varepsilon + \varepsilon 0)\left(a^2 (\varepsilon - \varepsilon 0) - b^2 (\varepsilon + \varepsilon 0)\right)},\ C \to \frac{b^2\, E0\, (\varepsilon - \varepsilon 0)}{\varepsilon + \varepsilon 0}\right\}\right\}$