4.7a

pt. 1

$$\rho(\vec{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

40/40

6

$$\Phi(\vec{r}) = \frac{1}{\varepsilon_0} \sum_{l,m} \frac{1}{2l+1} \left[\iiint Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{r}') r'^2 \sin\theta' dr' d\theta' d\phi' \right] \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

But invariant in ϕ, so $m=0$.

$$\Phi(\vec{r}) = \frac{1}{\varepsilon_0} \sum_{l} \frac{1}{2l+1} \left[\iiint Y^*_{l0}(\theta', \phi') r'^l \rho(\vec{r}') r^2 \sin\theta' dr' d\theta' d\phi' \right] \frac{Y_{l0}(\theta,\phi)}{r^{l+1}}$$

using 3.57, $Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)\ldots$

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \left[\iiint P_l(\cos\theta') r'^l \rho(\vec{r}') r'^2 \sin\theta' dr' d\theta' d\phi' \right] \frac{P_l(\cos\theta)}{r^{l+1}}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{r^{l+1}} \left[2\pi \int_0^{\pi}\int_0^{\infty} P_l(\cos\theta') \sin\theta' \, r'^{l+2} \left(\frac{1}{64\pi} r'^2 e^{-r'} \sin\theta' \right) dr' d\theta' \right.$$

$$= \frac{1}{128\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{r^{l+1}} \underbrace{\int_0^{\infty} r'^{l+4} e^{-r'} dr'}_{} \int_0^{\pi} P_l(\cos\theta') \sin^2\theta' \, d(\cos\theta')$$

use $\int_0^{\infty} k^n e^{-k} dk = n!$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - 1 = \sin^2\theta$

$\frac{2}{3}(P_0 - P_2)$

$\frac{2}{3} + \frac{1}{3} - \cos^2\theta$

$1 - \cos^2\theta$

$$= \frac{1}{128\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{r^{l+1}} (l+4)! \int_0^{\pi} P_l(\cos\theta')(1-\cos^2\theta') \, d(\cos\theta')$$

Now, $P_2(\cos\theta') = -\frac{1}{2} + \frac{3}{2}\cos^2\theta'$, and $P_0(\cos\theta') = 1$

Thus $\frac{2}{3} P_2(\cos\theta') + \frac{1}{3} P_0(\cos\theta') = \cos^2\theta'$

$$\cancel{\int_0^{\pi} P_l(\cos\theta') d\theta' - \frac{2}{3}\int_0^{\pi} P_l(\cos\theta') P_2(\cos\theta') d\theta'} - \frac{1}{3}\int_0^{\pi} P_l(\cos\theta') P_0(\cos\theta') d\theta'$$

$$\int_0^{\pi} P_l(\cos\theta') \frac{2}{3} \left(P_0(\cos\theta') - P_2(\cos\theta') \right) d(\cos\theta')$$

norm constant.

$$= \frac{2}{3}\int_0^{\pi} P_l P_0 \, dx - \frac{2}{3}\int_0^{\pi} P_l P_2 \, dx = \frac{2}{3}\delta_{l,0}\left(\frac{2}{2(0)+1}\right) - \frac{2}{3}\delta_{l,2}\left(\frac{2}{2(2)+1}\right)$$

$$= \frac{1}{128\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{2}{3} \frac{P_l(\cos\theta)}{r^{l+1}} (l+4)! \left(2\delta_{l,0} - \frac{2}{5}\delta_{l,2} \right) = \frac{1}{128\pi\varepsilon_0} \frac{2}{3} \left(\frac{2 P_0(\cos\theta)}{r} 4! - \frac{2 P_2(\cos\theta)}{5 r^3} 6! \right)$$

Note $\cos^2\theta = \cos(2\theta)$

$$= \frac{4}{3} \frac{1}{128\pi\varepsilon_0} \left(\frac{24}{r} - \frac{720}{5 r^3} \left(-\frac{1}{2} + \frac{3}{2}\cos^2\theta \right) \right)$$

$$\Phi(\vec{r}) = \Phi(r,\theta) = \boxed{\frac{1}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{6}{r^3}\left(\cos^2\theta \frac{3}{2} - \frac{1}{2} \right) \right)} = \boxed{\frac{1}{4\pi\varepsilon_0} \left(\frac{P_0(\cos\theta)}{r} - \frac{6 P_2(\cos\theta)}{r^3} \right)}$$

So long as we are "outside" the charge density, this is exact.

pt. 2.

Now recall, $Y_{\ell,0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$, so $P_0(\cos\theta) = Y_{00}\sqrt{\frac{4\pi}{2\ell+1}} = Y_{00}\sqrt{\frac{4\pi}{1}}$

$$P_2(\cos\theta) = Y_{20}\sqrt{\frac{4\pi}{5}}$$

$$\Phi(r,\theta) = \frac{1}{\varepsilon_0}\left[\frac{1\,P_0}{4\pi r} - \frac{6\,P_2}{4\pi r^3}\right] = \frac{1}{\varepsilon_0}\left[\frac{1}{\sqrt{4\pi}}\frac{Y_{00}}{r} - \frac{6}{\sqrt{20\pi}}\frac{Y_{20}}{r^3}\right]$$

$$\cancel{q_{00} = \sqrt{\frac{1}{4\pi}}} \qquad \cancel{q_{20} = \sqrt{\frac{36}{20\pi}}}$$

$$\frac{1}{\varepsilon_0}\left[\underbrace{\frac{1}{\sqrt{\pi 4}}}_{q_{00}} \cdot \underset{\uparrow\ \frac{1}{2\ell+1}\big|_{\ell=0}}{1}\ \frac{Y_{00}}{r} - \underbrace{\frac{30}{\sqrt{20\pi}}}_{q_{20}}\ \underset{\uparrow\ \frac{1}{2\ell+1}\big|_{\ell=0}}{\frac{1}{5}}\ \frac{Y_{20}}{r^3}\right]$$

$$\boxed{q_{00} = \sqrt{\frac{1}{4\pi}} \qquad q_{20} = 3\sqrt{\frac{5}{\pi}}}$$

(b) $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \longrightarrow \Phi(r,\theta) = \frac{1}{2\varepsilon_0}\int_0^\pi\int_0^\infty \frac{\left(\frac{1}{64\pi} r'^2 e^{-r'}\sin^2\theta'\right)}{|\vec{r}-\vec{r}'|}\, r'^2 dr' \sin\theta' d\theta'$

$$\frac{1}{|\vec{r}-\vec{r}'|} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

Azimuthaly symmetric $\Rightarrow m \to 0$. by 3.57 $\quad \frac{1}{|\vec{r}-\vec{r}'|} = \cancel{4\pi}\sum_{\ell=0}^{\infty} \cancel{\frac{1}{2\ell+1}}\frac{r_<^\ell}{r_>^{\ell+1}}\cancel{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta') P_\ell(\cos\theta)$

Thus $\Phi(r,\theta) = \frac{1}{128\pi\varepsilon_0}\int_0^\pi\int_0^\infty \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}} r'^2 e^{-r'}\sin^2\theta'\, P_\ell(\cos\theta') P_\ell(\cos\theta)\, r'^2 dr' \sin\theta' d\theta'$

for $r > r'$ (far away)

$$\Phi(r,\theta) = \frac{1}{128\pi\varepsilon_0}\int_0^\pi\int_0^\infty \sum_{\ell=0}^{\infty} \frac{(r')^{\ell+4} e^{-r'}}{r^{\ell+1}} dr'\, P_\ell(\cos\theta')\sin^3\theta'\, P_\ell(\cos\theta)\, d\theta'$$

amazing! same integral as in part a... $= \frac{1}{4\pi\varepsilon_0}\left(\frac{P_0(\cos\theta)}{r} - \frac{6\,P_2(\cos\theta)}{r^3}\right)$

for $r < r'$ (close to origin).

$$\Phi(r,\theta) = \frac{1}{128\pi\varepsilon_0}\int_0^\pi\int_0^\infty \sum_{\ell=0}^{\infty} \frac{r^\ell e^{-r'}}{(r')^{\ell-3}} dr'\, P_\ell(\cos\theta')\sin^3\theta'\, P_\ell(\cos\theta)\, d\theta'$$

$$= \frac{1}{128\pi\varepsilon_0}\sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta)(3-\ell)! \int_0^\pi P_\ell(\cos\theta')\sin^2\theta'\, d(\cos\theta') \to = \frac{2}{3}\left(2\,\delta_{\ell,0} - \frac{2}{5}\delta_{\ell,2}\right)$$

$$= \frac{1}{128\pi\varepsilon_0}\frac{4}{3}\left(\underset{1}{P_0(\cos\theta)}\, 6 - \frac{1}{5} r^2 P_2(\cos\theta)\right)$$

$$= \frac{1}{128\pi\varepsilon_0}\frac{4}{3}\left(6 - \frac{r^2}{5}P_2(\cos\theta)\right)$$

$$\boxed{\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right)}$$

both $r'<r$ and $r'>r$ cases contribute:

$$\Phi = \frac{1}{128\pi\varepsilon_0}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} r'^4 e^{-r'} \frac{2}{3}\left(2\delta_{l,0} - \frac{2}{5}\delta_{l,2}\right)dr' \; P_l(\cos\theta)$$

$$= \frac{1}{64\pi\varepsilon_0}\left(\frac{2}{3}\right)\left[\int_0^{\infty}\frac{r_<^0}{r_>^1} r'^4 e^{-r'}dr' - \frac{1}{5}\int_0^{\infty}\frac{r_<^2}{r_>^3} r'^4 e^{-r'}dr'\right]$$

$$= \frac{1}{64\pi\varepsilon_0}\left(\frac{2}{3}\right)\left[\int_r^{\infty} r'^3 e^{-r'}dr' + \int_0^r \frac{r'^4}{r}e^{-r'}dr' - \underbrace{\frac{1}{5}\int_0^r \frac{r'^6}{r^3}e^{-r'}dr'}_{P_2(\cos\theta)} - \underbrace{\frac{1}{5}\int_r^{\infty} r^2 r' e^{-r'}dr'}_{P_2(\cos\theta)}\right]$$

$$= \frac{1}{64\pi\varepsilon_0}\left(\frac{2}{3}\right)\left[e^{-r}(6+r(6+r(3+r))) + \frac{1}{r}\left[24+e^{-r}(-24-r(24+r(12+r(4+r))))\right]\right]$$

$$-\frac{1}{5r^3}$$

Blagh... MMA time. (good idea!)

$$\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{1}{r} - \frac{e^{-r}}{24}\left(r^2+6r+\frac{24}{r}+18\right)\right)\right.$$

$$\left. + P_2(\cos\theta)\left(\frac{e^{-r}}{120}\left(5r^2+30r+120+\frac{360}{r}+\frac{720}{r^2}+\frac{720}{r^3}\right) - \frac{6}{r^3}\right)\right]$$

Now @ large distances, $e^{-r}\to 0$, so $\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{6}{r^3}P_2(\cos\theta)\right)$

as $r\to 0$ though... $e^{-r} = 1 - r + \frac{r^2}{2} - \frac{r^3}{6}\cdots$ (~~higher orders aren't~~ imp.).

$$\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{24}\left(1-r+\frac{r^2}{2}\right)\left(r^2+6r+\frac{24}{r}+18\right)\right)$$

$$+ P_2\left(\frac{1}{120}\left(1-r+\frac{r^2}{2}\right)\left(5r^2+30r+120+\frac{360}{r}+\frac{720}{r^2}+\frac{720}{r^3}\right)-\frac{6}{r^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{24}\left(r^2+6r+\frac{24}{r}+\cancel{18} - r^3 - 6r^2 - 24 - 18r + \frac{r^4}{2} + 3r^3 + 12r + 9r^2\right)\right.$$

$$+ P_2\left(\frac{1}{120}\left(5r^2+30r+\cancel{120}+\frac{360}{r}+\frac{720}{r^2}+\frac{720}{r^3} - 5r^3 - 30r^2 - 120r - \cancel{360} - \frac{720}{r} - \cancel{\frac{720}{r^2}}\right.\right.$$

$$\left.\left.+\frac{5}{2}r^4 + \cancel{15r^3} + 60r^2 + 180r + \cancel{360} + 360r\right) - \frac{6}{r^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{24}\left(4r^2+12r+\frac{24}{r}-6+\cancel{2r^3}+\cancel{\tfrac{1}{2}r^4}\right) + P_2\left(-\frac{6}{r^3}+\frac{1}{120}\left(35r^2+450r+120-\frac{360}{r}+\frac{720}{r^3}\right.\right.\right.$$

$$\left.\left.\left.+10r^3+\frac{5}{2}r^4\right)\right)\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\cancel{\frac{1}{r}} - \frac{1}{6}r^2 - \frac{1}{2}r - \cancel{\frac{1}{r}} + \frac{1}{4}\right) + P_2(\cos\theta)\left(\frac{7}{24}r^2 + \frac{15}{4}r + 1 - \frac{3}{r} - \cancel{\frac{6}{r^3}} + \cancel{\frac{6}{r^3}}\right)$$

Or MMA again, and neglect higher than r^2 terms...
it reduces to

$$\Phi(\vec{r}) \cong \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right]$$

10

Actually doing this integral is kind of annoying by hand...

$$\frac{2}{3}\frac{1}{64\,\pi\,\epsilon}\left(\int_r^\infty s^3\,e^{-s}\,ds+\frac{1}{r}\int_0^r s^4\,e^{-s}\,ds-\frac{1}{5\,r^3}\,pp\int_0^r s^6\,e^{-s}\,ds-\frac{r^2}{5}\,pp\int_r^\infty s^1\,e^{-s}\,ds\right)\;//\;\mathbf{FullSimplify}$$

$$\frac{1}{96\,\pi\,r^3\,\epsilon}\,e^{-r}\left(pp\left(144-144\,e^r+r\left(12+r^2\right)(12+r\,(6+r))\right)-r^2\left(24-24\,e^r+r\,(18+r\,(6+r))\right)\right)$$

Now for the expansion. We must multiply two terms out when seeking the potential at the origin. I'll do them seperatly.

$$\frac{1}{r}-\frac{1}{24}\left(1-r+\frac{r^2}{2}-\frac{r^3}{6}\right)*\left(r^2+6\,r+\frac{24}{r}+18\right)\;//\;\mathbf{FullSimplify}$$

$$\frac{1}{144}\left(36+r^3\,(6+r\,(3+r))\right)$$

This is just $\frac{36}{144}=\frac{1}{4}$ neglecting terms of order r^3 and greater.

We two more terms in the expanson this time to make it work...

$$\frac{-6}{r^3}+\frac{1}{120}\left(1-r+\frac{r^2}{2}-\frac{r^3}{6}+\frac{r^4}{4!}-\frac{r^5}{5!}\right)*\left(5\,r^2+30\,r+120+\frac{360}{r}+\frac{720}{r^2}+\frac{720}{r^3}\right)\;//\;\mathbf{FullSimplify}$$

$$-\frac{r^2\left(24+r\left(12+r^2\right)\left(2+r+r^2\right)\right)}{2880}$$

Not really in fully simple form yet. If you expand it out, the only term less than order 3 is $\frac{-r^2}{120}$.