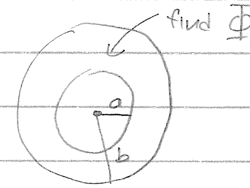
Jackson 3-27

(a) We have two concentric spheres with the ϕ-independent Electric fields are specified on the boundaries:



$E_r(r=b) = -E_0\cos(\theta)$ and $E_r(r=a)=0$.

We are looking for $\Phi(\vec{x})$ between shells.

Given: $G(\vec{x},\vec{x}') = \sum_{\ell=0}^{\infty} g_\ell(r,r')\,P_\ell(\cos\gamma)$. g_ℓ is specified. We'll get to it later.

we know $\frac{-\partial\Phi(\vec{x}')}{\partial n} = E_n$, so we have a Neumann Boundary condition.

Specifically: Inside $\frac{\partial\Phi(\vec{x}')}{\partial n'} = -E_r = 0$, Outside: $\frac{\partial\Phi(\vec{x}')}{\partial n} = \frac{\partial\Phi(\vec{x}')}{\partial(-r')} = -(-E_r) = E_r = -E_0\cos\theta$

Now for Neuman boundary conditions:

$$\Phi(\vec{x}') = \frac{-1}{4\pi}\sum_{\ell=0}^{\infty}\int_0^{2\pi}\int_0^{\pi} b^2\sin\theta'\,d\theta'\,d\phi'\,(-E_0\cos\theta)\,P_\ell(\cos\gamma)\,g_\ell(r,r')$$

(We only need OUTER integral).

$$= \frac{b^2E_0}{4\pi}\sum_{\ell=0}^{\infty} g_\ell(r,b)\int_0^{2\pi}\int_0^{\pi} P_\ell(\cos\gamma)\sin\theta'\cos\theta'\,d\theta'\,d\phi'$$

Sadly, we can't just integrate $P_\ell(\cos\gamma)$. Instead, use Eq. 3.62.

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

so $$\Phi(\vec{x}) = \frac{b^2E_0}{2\ell+1}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)\,g_\ell(r,b)\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell m}(\theta',\phi')\sin\theta'\cos\theta'\,d\theta'\,d\phi'$$

and note that $Y_{10}(\theta',\phi') = \sqrt{\frac{3}{4\pi}}\cos\theta' \Rightarrow \cos\theta' = \sqrt{\frac{4\pi}{3}}\,Y_{10}(\theta',\phi')$

so, by eq. 3.55, $\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell' m'}(\theta',\phi')\,Y_{\ell m}(\theta',\phi')\sin\theta\,d\theta\,d\phi = \delta_{\ell\ell'}\delta_{mm'}$

which implies $m=0$ (as we knew b/c of azimuthal symmetry, and $\ell=0$).

Now we know: $\Phi(\vec{x}) = \frac{b^2E_0}{3}Y_{10}(\theta,\phi)\,g_1(r,b)\sqrt{\frac{4\pi}{3}} = \frac{b^2E_0}{3}g_1(r,b)\cos(\theta)$.

From problem 3.26, $g_1(r,b) = \frac{1}{b^3-a^3}\left[2br + \frac{1}{2}\frac{a^3b^3}{r^2b^2} + \frac{a^3r}{b^2} + \frac{a^3b}{r^2}\right] + \frac{r}{b^2}$

where I have used $b=r'=r_>$ and $r=r_<$.

Simplifying, using Mathematica (see attached). $g_1(r,b) = \frac{3}{2}b\,\frac{(a^3+2r^3)}{(b^3-a^3)r^2}$

Plugging it in:

$$\Phi(\vec{x}) = \frac{E_0b^3}{2}\frac{(a^3+2r^3)}{(b^3-a^3)r^2}\cos\theta = \frac{E_0}{2}\frac{(a^3+2r^3)}{(1-p^3)r^2}\cos\theta$$

where $p=\frac{a}{b}$.

Finally: $\Phi(\vec{x}) = \Phi(r,\theta) = \boxed{\frac{E_0\,r\cos\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)}$ ✓

Now $\vec{E}_\theta = \frac{-\partial}{\partial\theta}\Phi(r,\theta) = -\left(\frac{-E_0 r\sin\theta}{1-p^3}\right)\left(1+\frac{a^3}{2r^3}\right) \Rightarrow \boxed{\vec{E}_\theta = \frac{E_0\sin\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)}$ ✓

$$\vec{E}_r = -\frac{\partial}{\partial r}\Phi(r,\theta) = -\left(E_0 \frac{\cos\theta}{1-p^3} + \frac{-E_0\cos\theta}{1-p^3}\cdot\frac{a^3}{r^3}\right)$$

$$\boxed{\vec{E}_r = \frac{-E_0\cos\theta}{1-p^3}\left(1-\frac{a^3}{r^3}\right)}\ \checkmark$$

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In[268]:= $\left(\frac{1}{\left(b^3-a^3\right)}\left(2\,b\,r+\frac{1}{2}\,\frac{a^3\,b^3}{r^2\,b^2}+\frac{a^3\,r}{b^2}+\frac{a^3\,b}{r^2}\right)+\frac{r}{b^2}\right)$ // FullSimplify

Out[268]= $\frac{3\,b\,\left(a^3+2\,r^3\right)}{2\,\left(-a^3+b^3\right)\,r^2}$