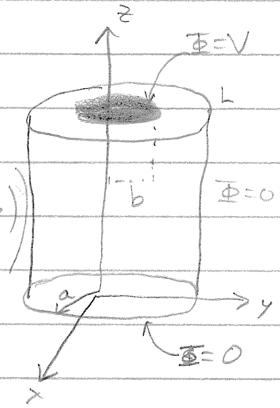
For a point charge $\Phi(\vec{x},\vec{x}') = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{x}-\vec{x}'|}$ Thus we divide

Jackson 3-24 through for the given potentials by $q/4\pi\varepsilon_0$ to get $G(\vec{x},\vec{x}')$

Also note for this situation that we have azimuthal symmetry, so only the $m=0$ case remains.



① $$G(\vec{x},\vec{x}') = \frac{4}{a}\sum_{n=1}^{\infty}\frac{J_0(x_n\rho/a)\,J_0(x_n\rho'/a)}{x_n J_1^2(x_n)\sinh(x_n L/a)}\cdot\sinh\left(\frac{x_n z_<}{a}\right)\sinh\left(\frac{x_n}{a}(L-z_>)\right)$$

② $$G(\vec{x},\vec{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_0\left(\frac{n\pi\rho_<}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}$$
$$\times\left[I_0\left(\frac{n\pi a}{L}\right)K_0\left(\frac{n\pi\rho_>}{L}\right) - K_0\left(\frac{n\pi a}{L}\right)I_0\left(\frac{n\pi\rho_>}{L}\right)\right]$$

③ $$G(\vec{x},\vec{x}') = \frac{8}{La^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right)J_0\left(\frac{x_n\rho}{a}\right)J_0\left(\frac{x_n\rho'}{a}\right)}{\left[(x_n/a)^2+(k\pi/L)^2\right]J_1^2(x_n)}$$

Now we get the potential in the cylinder using:

$$\Phi(\vec{x}) = \underbrace{\cancel{\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,dV}}_{0 \text{ No bulk charge}} - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}\,da'$$

To find the normal derivative of G, we differentiate w/ respect to z' @ $z'=L$

① First note $z_< = z$, and $z_> = z'$, then $\frac{\partial}{\partial z'}\Big|_{z'=L}\sinh\left(\frac{x_n}{a}(L-z')\right) = -\frac{x_n}{a}\cosh\left(\frac{x_n}{a}(L-L)\right) = -\frac{x_n}{a}$

$$\frac{\partial G}{\partial z'} = -\frac{4}{a^2}\sum_{n=1}^{\infty}\frac{J_0(x_n\rho/a)\,J_0(x_n\rho'/a)}{J_1^2(x_n)\sinh\left(\frac{x_n L}{a}\right)}\cdot\sinh(x_n z/a)$$

② Note that $\frac{\partial}{\partial z'}\sin\left(\frac{n\pi z'}{L}\right)\Big|_{z'=L} = \frac{n\pi}{L}\cos\left(\frac{n\pi L}{L}\right) = \frac{n\pi}{L}\cos(n\pi) = \frac{n\pi}{L}(-1)^n$

$$\frac{\partial G}{\partial z'} = \frac{4\pi}{L^2}\sum_{n=1}^{\infty}\frac{n(-1)^n\sin\left(\frac{n\pi z}{L}\right)I_0\left(\frac{n\pi\rho_<}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}\cdot\left[I_0\left(\frac{n\pi a}{L}\right)K_0\left(\frac{n\pi\rho_>}{L}\right) - K_0\left(\frac{n\pi a}{L}\right)I_0\left(\frac{n\pi\rho_>}{L}\right)\right]$$

③ Note that $\frac{\partial}{\partial z'}\sin\left(\frac{k\pi z'}{L}\right)\Big|_{z'=L} = \frac{k\pi}{L}(-1)^k$

$$\frac{\partial G}{\partial z'} = \frac{8\pi}{L^2a^2}\sum_{k,n=1}^{\infty}\frac{k(-1)^k\sin\left(\frac{k\pi z}{L}\right)J_0\left(\frac{x_n\rho}{a}\right)J_0\left(\frac{x_n\rho'}{a}\right)}{\left[(x_n/a)^2+(k\pi/L)^2\right]J_1^2(x_n)}$$

Now we will integrate the source potential with these three kernels over the shaded disk in the figure.

$$\Phi(\rho,z) = -\frac{1}{4\pi}\int_0^b\int_0^{2\pi} V\frac{\partial G}{\partial z'}\rho'\,d\phi'\,d\rho' = -\frac{V}{2}\int_0^b\frac{\partial G}{\partial z'}\rho'\,d\rho'$$

① $\Phi(\rho,z) = \frac{2V}{a^2}\sum_{n=1}^{\infty} \frac{J_0(x_n\rho/a)\sinh(x_n z/a)}{J_1^2(x_n)\sinh(\frac{x_n L}{a})}\int_0^b J_0(x_n\rho'/a)\rho' d\rho'$

See attached Mathematica. $\int_0^b J_0(\gamma\rho')\rho' d\rho' = \frac{b}{\gamma}J_1(b\gamma) = \frac{ba}{x_n}J_1\left(\frac{x_n b}{a}\right)$

$$\boxed{\Phi(\rho,z) = \frac{2Vb}{a}\sum_{n=1}^{\infty} \frac{J_0\left(\frac{x_n\rho}{a}\right)\sinh\left(\frac{x_n z}{a}\right)J_1\left(\frac{x_n b}{a}\right)}{x_n J_1^2(x_n)\sinh\left(\frac{x_n L}{a}\right)}}$$

② $\Phi(\rho,z) = \frac{-2\pi V}{L^2}\int_0^b \sum_{n=1}^{\infty} \frac{n(-1)^n \sin\left(\frac{n\pi z}{L}\right) I_0(\pi n\rho_</L)}{I_0\left(\frac{n\pi a}{L}\right)}\left[\left(I_0\left(\frac{n\pi a}{L}\right)K_0\left(\frac{n\pi\rho_>}{L}\right) - K_0\left(\frac{n\pi a}{L}\right)I_0\left(\frac{n\pi\rho_>}{L}\right)\right]\rho' d\rho'$

Sadly, there are two regions. First:

for $\rho_> = \rho$, $\rho_< = \rho'$ (i.e @ large radii)

$= \frac{-2\pi V}{L^2}\sum_{n=1}^{\infty} \frac{n(-1)^n \sin\left(\frac{n\pi z}{L}\right)\left[I_0\left(\frac{n\pi a}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right) - K_0\left(\frac{n\pi a}{L}\right)I_0\left(\frac{n\pi\rho}{L}\right)\right]}{I_0\left(\frac{n\pi a}{L}\right)}\int_0^b I_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho'$

By Mathematica again $\int_0^b I_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho' = \frac{bL}{n\pi}I_1\left(\frac{n\pi b}{L}\right)$

$$\boxed{\Phi_{\rho>b}(\rho,z) = \frac{-2bV}{L}\sum_{n=1}^{\infty} \frac{(-1)^n \sin\left(\frac{n\pi z}{L}\right)I_1\left(\frac{n\pi b}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}\left(I_0\left(\frac{n\pi a}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right) - K_0\left(\frac{n\pi a}{L}\right)I_0\left(\frac{n\pi\rho}{L}\right)\right)}$$

for $\rho_< = \rho$, $\rho_> = \rho'$ (i.e. @ small radii $< b$).

$= \frac{-2\pi V}{L^2}\sum_{n=1}^{\infty} \frac{n(-1)^n \sin\left(\frac{n\pi z}{L}\right)I_0\left(\frac{n\pi\rho}{L}\right)I_0\left(\frac{n\pi a}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}\int_0^b K_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho'$

$+ \frac{2\pi V}{L^2}\sum_{n=1}^{\infty} \frac{n(-1)^n \sin\left(\frac{n\pi z}{L}\right)I_0\left(\frac{n\pi\rho}{L}\right)K_0\left(\frac{n\pi a}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}\int_0^b I_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho'$

The first integral, by Mathematica: $\int_0^b K_0\left(\frac{n\pi\rho'}{L}\right)\rho' d\rho' = \frac{L}{n^2\pi^2}\left(L - bn\pi K_1\left(\frac{n\pi b}{L}\right)\right)$

The second is the same boring one from before. $\frac{bL}{n\pi}I_1\left(\frac{n\pi b}{L}\right)$

$$\boxed{\Phi_{\rho<b}(\rho,z) = \frac{-2V}{L}\sum_{n=1}^{\infty} \frac{(-1)^n \sin\left(\frac{n\pi z}{L}\right)I_0\left(\frac{n\pi\rho}{L}\right)}{I_0\left(\frac{n\pi a}{L}\right)}\left(\frac{1}{n\pi}I_0\left(\frac{n\pi a}{L}\right)\left(L - bn\pi K_1\left(\frac{n\pi b}{L}\right)\right) - K_0\left(\frac{n\pi a}{L}\right)I_1\left(\frac{n\pi b}{L}\right)\right)}$$

③ $\Phi(\rho,z) = \frac{-4\pi V}{L^2 a^2}\sum_{k,n=1}^{\infty} \frac{k(-1)^k \sin\left(\frac{k\pi z}{L}\right)J_0\left(\frac{x_n\rho}{a}\right)}{\left(\frac{x_n^2}{a^2} + \left(\frac{k\pi}{L}\right)^2\right)J_1(x_n)}\int_0^b J_0\left(\frac{x_n\rho'}{a}\right)\rho' d\rho'$ Same as in ①

$$\boxed{\Phi(\rho,z) = \frac{-4\pi bV}{L^2 a}\sum_{k,n=1}^{\infty} \frac{k(-1)^k \sin\left(\frac{k\pi z}{L}\right)J_0\left(\frac{x_n\rho}{a}\right)J_1\left(\frac{x_n b}{a}\right)}{x_n\left(\frac{x_n^2}{a^2} + \left(\frac{k\pi}{L}\right)^2\right)J_1(x_n)}}$$

fun! ☺

(10)

$$\left\{\int_0^b \text{BesselJ}[0, \gamma*\rho]\,\rho\,d\rho,\ \int_0^b \text{BesselI}[0, \gamma*\rho]\,\rho\,d\rho,\ \int_0^b \text{BesselK}[0, \gamma*\rho]\,\rho\,d\rho\right\}$$

$$\left\{\frac{b\,\text{BesselJ}[1, b\gamma]}{\gamma},\ \frac{b\,\text{BesselI}[1, b\gamma]}{\gamma},\ \frac{1 - b\gamma\,\text{BesselK}[1, b\gamma]}{\gamma^2}\right\}$$