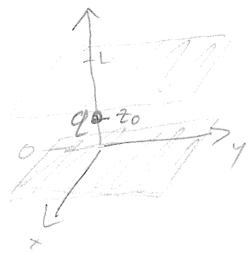
4 Jackson 3.20

ⓐ $$G(\vec{x},\vec{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)$$

The potential can again be found...

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da'$$



The charge Distr'o is

$\rho(\vec{x}') = A\,\delta(z'-z_0)\,\delta(\rho')$ for a pt. charge @ $z_0$. ($\Phi(\vec{x}')=0$ is given)

(invariant in $\phi$)

Integrate to get $A$:

$$\int \rho(\vec{x})\,dV = q = A\int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\delta(z'-z_0)\,\delta(\rho')\rho'\,dz'\,d\rho'\,d\phi' = A2\pi\rho' \Rightarrow A = \frac{q}{2\pi\rho'}$$

Thus $\rho(\vec{x}') = \dfrac{q}{2\pi\rho'}\delta(z'-z_0)\,\delta(\rho')$

the coordinate.

Set the charge to be on the z-axis. Then $\rho'=0$.

Obviously then $\rho'$ is $\rho_<$ (so $\rho$ must be $\rho_>$).

$I_m\left(\frac{n\pi}{L}\rho_<\right) = I_m(0) = 0$ for all $m$. ( ).

Now we can integrate $\rho(\vec{x}')$ w/ $G(\vec{x},\vec{x}')$ to get $\Phi(\vec{x})$ which is what we seek.

$$\Phi(\vec{x}) = \Phi(\rho,z) = \frac{1}{4\pi\varepsilon_0}\int_0^{\infty}\int_0^{L}\int_0^{2\pi} d\phi'\,dz'\,\cancel{\rho}\,d\rho\left(\frac{q}{2\pi\cancel{\rho'}}\delta(z'-z_0)\,\delta(\rho')\right)\frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)K_m\left(\frac{n\pi}{L}\rho\right)\right]$$

$$= \frac{q}{2\pi^2\varepsilon_0 L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)K_m\left(\frac{n\pi}{L}\rho\right)\int_0^{\infty}\int_0^{2\pi}\delta(\rho')\,e^{im(\phi-\phi')}\,d\phi'\,d\rho'\right]$$

$$= \frac{q}{2\pi^2\varepsilon_0 L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)K_m\left(\frac{n\pi}{L}\rho\right)e^{im\phi}\int_0^{2\pi}e^{-im\phi'}\,d\phi'\right.$$

Now, if $m=0$, $\int_0^{2\pi}e^{-im\phi'}\,d\phi' = \int_0^{2\pi}d\phi' = 2\pi$

if $m\neq 0$, $\int_0^{2\pi}e^{-im\phi'} = \frac{i}{m}e^{-im\phi'}\Big|_0^{2\pi} = \frac{i}{m}\cancel{e^{-im2\pi}} - \frac{i}{m} = \frac{i}{m} - \frac{i}{m} = 0$

Thus $m=0$ is the only case that matters.

$$\boxed{\Phi(\rho,z) = \frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)}$$

Azimuthally symmetric, as expected.

ⓑ $$\sigma_0(\rho) = -\varepsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=0} = \frac{q}{\pi L}\sum_{n=1}^{\infty}\frac{\partial}{\partial z}\left(\sin\left(\frac{n\pi z}{L}\right)\right)\bigg|_{z=0}\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

$$\to \frac{n\pi}{L}\cos\left(\frac{n\pi z}{L}\right)\bigg|_{z=0} = \frac{n\pi}{L}\cos(0) = \frac{n\pi}{L}$$

$$\boxed{\sigma_0(\rho) = \frac{q}{L^2}\sum_{n=1}^{\infty}n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)}$$

$$\sigma_L(\rho) = -\varepsilon_0\frac{\partial\Phi}{\partial z}\bigg|_{z=L} = \frac{q}{\pi L}\sum_{n=1}^{\infty}\frac{\partial}{\partial z}\left(\sin\left(\frac{n\pi z}{L}\right)\right)\bigg|_{z=L}\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

→ Comparison to results from 3.19?

$$\boxed{\sigma_L(\rho) = \frac{q}{L^2}\sum_{n=1}^{\infty}n(-1)^n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)}$$

$$\to \frac{n\pi}{L}\cos\left(\frac{n\pi z}{L}\right)\bigg|_{z=L} = \frac{n\pi}{L}\cos(n\pi) = \frac{n\pi}{L}(-1)^n$$