(Jackson 3-13) Use green fn. 3.125

Then $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \cancel{\rho(\vec{x}')G(\vec{x},\vec{x}')d^3x'} - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}da'$ (3.126)

$\rightarrow 0$ (no charge)

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r'} = \frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-(a/b)^{2l+1}\right)}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

Now on inner surface $r'=a$, we are interested only in r greater than this boundary. So: inner: $r_< = r'$, $r_> = r \Rightarrow \left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)$

outer: $r_< = r$, $r_> = r' \Rightarrow \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)$

$$\text{inner: } -\left.\frac{\partial G}{\partial r'}\right|_{r'=a} = -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-(a/b)^{2l+1}\right)}\left(\underbrace{l a^{(l-1)} + (l+1)\frac{a^{2l+1}}{a^{(l+2)}}}_{\Rightarrow\ l a^{l-1} + (l+1)a^{l-1} = (2l+1)a^{l-1}}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)$$

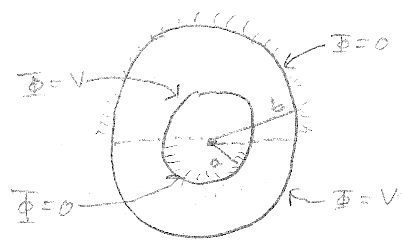
$$= -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)a^{l-1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)$$

$$\text{outer: } \left.\frac{\partial G}{\partial r'}\right|_{r'=b} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-(a/b)^{2l+1}\right)}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\underbrace{\left(\frac{-(l+1)}{b^{l+2}} - \frac{l b^{l-1}}{b^{2l+1}}\right)}$$

$$= -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{b^{l+2}\left(1-(a/b)^{2l+1}\right)}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) \qquad \frac{-l-1}{b^{l+2}} - \frac{l}{b^{l+2}} = \frac{-2l-1}{b^{l+2}}$$

inner: $\Phi_{a,up}(\vec{x}') = V$, $\Phi_{a,down}(\vec{x}') = 0$

outer: $\Phi_{b,up}(\vec{x}') = 0$, $\Phi_{b,down}(\vec{x}') = V$



All we have to do now is integrate these source potentials w/ the appropriate normal derivatives in those regions. The only regions where that integral isn't zero is the bottom of the outer sphere, and the top of the inner sphere

Note that there's azimuthal symmetry, so $m=0$. $Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$ (3.57)

$$\frac{\partial G}{\partial r'_{in}} = \sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta')P_l(\cos\theta)a^{l-1}}{\left(1-(a/b)^{2l+1}\right)}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \qquad \frac{\partial G}{\partial r'_{out}} = -\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta)P_l(\cos(\theta))}{b^{l+2}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)$$

$$\Phi(r,\theta) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} V\left.\frac{\partial G}{\partial r'}\right|_{r'=a}\overset{r^2}{a^2}\sin\theta' d\theta' d\phi' - \frac{1}{4\pi}\int_0^{2\pi}\int_{\pi/2}^{\pi} V\left.\frac{\partial G}{\partial r'}\right|_{r'=b} b^2\sin\theta' d\theta' d\phi'$$

$$= \frac{-V}{2}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(-\int_0^{\frac{\pi}{2}} a^{l+1}P_l(\cos\theta')\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\sin\theta' d\theta' + \int_{\pi/2}^{\pi}\frac{P_l(\cos\theta')}{b^l}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\sin\theta' d\theta'\right)$$

$$= \frac{V}{2}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(\int_0^{\pi/2}P_l(\cos\theta')\sin\theta' d\theta'\left(\frac{a^{l+1}}{r^{l+1}} - \frac{r^l a^{l+1}}{b^{2l+1}}\right) - \int_{\pi/2}^{\pi}P_l(\cos\theta')\sin\theta' d\theta'\left(\frac{r^l}{b^l} - \frac{a^{2l+1}}{b^l r^{l+1}}\right)\right)$$

✓ To Mathematica... See attached output for these integrals.

$$\int_0^{\pi/2}P_l(\cos\theta')\sin\theta' d\theta' = \frac{\sqrt{\pi}}{2\Gamma(1-l/2)\Gamma\left(\frac{3+l}{2}\right)} = -\int_{\pi/2}^{\pi}P_l(\cos\theta')\sin\theta' d\theta'$$

$\rightarrow$

Also note that all the even terms $\ell \geq 2$ are equal to zero in these integrals. (shown in Mathematica as well).

$$= \frac{V}{2} \sum_{\ell=0,1,3,5,7,\ldots} \frac{(2\ell+1) P_\ell(\cos\theta)\sqrt{\pi}}{\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right) 2\Gamma\left(1-\frac{\ell}{2}\right)\Gamma\left(\frac{3+\ell}{2}\right)} \left(\frac{a^{\ell+1}}{r^{\ell+1}} - \frac{r^\ell a^{\ell+1}}{b^{2\ell+1}} + \frac{r^\ell}{b^\ell} - \frac{a^{2\ell+1}}{b^\ell r^{\ell+1}}\right)$$

multiply top and bottom by $b^{2\ell+1}$

$$= \frac{V}{2} \sum \frac{(2\ell+1) P_\ell(\cos\theta)\sqrt{\pi}}{2\Gamma\left(1-\frac{\ell}{2}\right)\Gamma\left(\frac{3+\ell}{2}\right)} \left(\frac{1}{b^{2\ell+1}-a^{2\ell+1}}\right)\left(\frac{a^{\ell+1} b^{2\ell+1}}{r^{\ell+1}} - \underbrace{\frac{r^\ell a^{\ell+1} b^{2\ell+1}}{b^{2\ell+1}}}_{r^\ell a^{\ell+1}} + \underbrace{\frac{r^\ell b^{2\ell+1}}{b^\ell}}_{r^\ell b^{\ell+1}} - \underbrace{\frac{a^{2\ell+1} b^{2\ell+1}}{b^\ell r^{\ell+1}}}_{\frac{a^{2\ell+1} b^{\ell+1}}{r^{\ell+1}}}\right)$$

$$= \frac{V}{2} \sum_{\ell=0,1,3,5,7,\ldots}^{\infty} \frac{(2\ell+1) P_\ell(\cos\theta)\sqrt{\pi}}{2\Gamma\left(1-\frac{\ell}{2}\right)\Gamma\left(\frac{3+\ell}{2}\right)} \left(\frac{a^{\ell+1}b^{2\ell+1} - r^{2\ell+1} a^{\ell+1} b^{2\ell+1} + r^{2\ell+1} b^{2\ell+1} - a^{2\ell+1} b^{\ell+1}}{r^{\ell+1}\left(b^{2\ell+1}-a^{2\ell+1}\right)}\right)$$

$$\boxed{\Phi(r,\theta) = \frac{V}{2} \sum_{\ell=0,1,3,5,7,\ldots}^{\infty} \frac{(2\ell+1) P_\ell(\cos\theta)\sqrt{\pi}}{2\Gamma\left(1-\frac{\ell}{2}\right)\Gamma\left(\frac{3+\ell}{2}\right)} \left(\frac{a^{\ell+1} b^{\ell+1}\left(a^\ell - b^\ell\right) + r^{2\ell+1} b^{2\ell+1}\left(1-a^{\ell+1}\right)}{r^{\ell+1}\left(b^{2\ell+1}-a^{2\ell+1}\right)}\right)}$$

How do your results compare to those obtained from the differential equation? What should you expect? 8

In[198]:= $\int_0^{\frac{\pi}{2}}$ **LegendreP[k, Cos[θ]] * Sin[θ] dθ**

Out[198]= $\frac{\sqrt{\pi}}{2\,\text{Gamma}\left[1-\frac{k}{2}\right]\,\text{Gamma}\left[\frac{3+k}{2}\right]}$

In[189]:= $\int_{\frac{\pi}{2}}^{\pi}$ **LegendreP[k, Cos[θ]] * Sin[θ] dθ**

Out[189]= $-\frac{\sqrt{\pi}}{2\,\text{Gamma}\left[1-\frac{k}{2}\right]\,\text{Gamma}\left[\frac{3+k}{2}\right]} + \frac{2\,\text{Sin}[k\,\pi]}{\left(k+k^2\right)\pi}$

(*But note that Sin[k*π] is always zero above, so we dropped the term*)

In[209]:= **Table[** $\int_0^{\frac{\pi}{2}}$ **LegendreP[k, Cos[θ]] * Sin[θ] dθ, {k, 0, 12, 1}]**

Table[$\int_{\frac{\pi}{2}}^{\pi}$ **LegendreP[k, Cos[θ]] * Sin[θ] dθ, {k, 0, 12, 1}]**

Out[209]= $\left\{1, \frac{1}{2}, 0, -\frac{1}{8}, 0, \frac{1}{16}, 0, -\frac{5}{128}, 0, \frac{7}{256}, 0, -\frac{21}{1024}, 0\right\}$

Out[210]= $\left\{1, -\frac{1}{2}, 0, \frac{1}{8}, 0, -\frac{1}{16}, 0, \frac{5}{128}, 0, -\frac{7}{256}, 0, \frac{21}{1024}, 0\right\}$