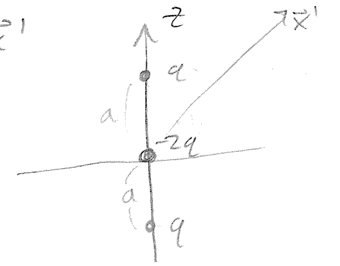
B Jackson 3.7

(a) $\Phi(x,y,z) = \frac{kq}{|\vec{r}-a\hat{z}|} + \frac{kq}{|\vec{r}+a\hat{z}|} - \frac{2kq}{|\vec{r}|} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-a\hat{z}|} + \frac{1}{|\vec{r}+a\hat{z}|} - \frac{2}{|\vec{r}|}\right]$ is the Green's Function

we expand w/ Legendre Polynomials: (using 3.38) (b/c no ϕ-dependence).

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

γ is $\angle$ btwn. $\vec{x}$ an $\vec{x}'$



when $|\vec{r}| < a$ $\quad \frac{1}{|\vec{r}\pm a\hat{z}|} = \sum_{l=0}^{\infty} \frac{r^l}{(\pm a)^{l+1}} P_l(\cos\gamma)$

when $|\vec{r}| > a$ $\quad \frac{1}{|\vec{r}\pm a\hat{z}|} = \sum_{l=0}^{\infty} \frac{(\pm a)^l}{r^{l+1}} P_l(\cos\gamma)$

NO ϕ-dep. so $\gamma \to \theta$

choose $|\vec{r}|>a$ (we want the case when $a\to 0$, which is like moving away).

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty} \frac{(-a)^l}{r^{l+1}} P_l(\cos\theta) + \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta) - \frac{2}{r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty}\left(\frac{(-a)^l}{r^{l+1}} P_l(\cos\theta) + \frac{a^l}{r^{l+1}} P_l(\cos\theta)\right) - \frac{2}{r}\right] = \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty}\left(\frac{P_l(\cos\theta)}{r^{l+1}}((-a)^l + a^l)\right) - \frac{2}{r}\right]$$

when l = odd, $(-a)^l + a^l = 0$, but when l = even $(-a)^l + a^l = 2a^l$

$$= \frac{2q}{4\pi\varepsilon_0}\left[\sum_{l=\text{even}}\left(\frac{P_l(\cos\theta)a^l}{r^{l+1}}\right) - \frac{1}{r}\right]$$

Now, note that in the $l=0$ case, the sumation term is just $\frac{1}{r}$

$$= \frac{q}{2\pi\varepsilon_0}\left[\sum_{l=2,4,6}^{\infty}\frac{P_l(\cos\theta)a^l}{r^{l+1}}\right]$$

Now Let $a\to 0$, so all the terms get small. $a^2 > a^4 > a^6 > a^8 > \cdots > a^\infty$, so the $l=2$ term dominates.

$$\Phi(r,\theta) \approx \frac{q}{2\pi\varepsilon_0}\,\frac{3\cos^2\theta - 1}{2}\,\frac{a^2}{r^3} = \frac{qa^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1) = \boxed{\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^3}(3\cos^2\theta - 1)}$$

(b) Add the spherical boundary at $r=b>a$. The potential there is zero.

There must be some other potential present such that @ the boundary $\Phi(r=b)=0$.
In general, we just add $\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$ to the potential, and apply Boundary Conditions. First note that Φ is finite as $r\to 0$, so $B_l = 0$ inside the sphere. (The only region we care about).

$$\Phi(r,\theta) = \frac{q}{2\pi\varepsilon_0}\left[\sum_{l=2,4,6,..}^{\infty}\frac{P_l(\cos\theta)a^l}{r^{l+1}}\right] + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

B.C.: $\Phi(b,\theta) = 0 \implies \frac{-q}{2\pi\varepsilon_0}\left[\sum_{l=2,4,6}^{\infty}\frac{P_l(\cos\theta)a^l}{b^{l+1}}\right] = \sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta)$

So for $l=2,4,6,\ldots$ $\quad \frac{-q}{2\pi\varepsilon_0}\frac{P_l(\cos\theta)a^l}{b^{l+1}} = A_l b^l P_l(\cos\theta)$

$\implies A_l = \frac{-q}{2\pi\varepsilon_0}\frac{a^l}{b^{2l+1}}$

Then for $l = 2,4,6,\ldots$

$$\Phi(r,\theta) = \frac{q}{2\pi\varepsilon_0}\left[\sum_{l=2,4,6..}^{\infty}\left[\frac{P_l(\cos\theta)a^l}{r^{l+1}} - \frac{a^l r^l}{b^{2l+1}}P_l(\cos\theta)\right]\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\sum_{l=2,4,6}^{\infty}\left[P_l(\cos\theta)a^l\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\right]\right]$$

again, as $a\to 0$, the $l=2$ term dominates.

→ What is the potential for ~~r>b~~ $r<a$?

$$\Phi(r,\theta) = \frac{q}{2\pi\varepsilon_0}\left(P_2(\cos\theta)a^2\left(\frac{1}{r^3} - \frac{r^2}{b^5}\right)\right) = \boxed{\frac{Q}{2\pi\varepsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right)P_2(\cos\theta)}$$

(8)