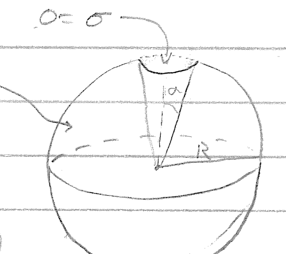
Jackson 3-2

ⓐ We are given the uniformly charged sphere of radius R w/ missing cap shown @ right. $\sigma = \frac{Q}{4\pi R^2}$



Goal: Find Φ_{in} and Φ_{out}.

In general (by 3.33), $\Phi(r,\theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta)$

we know Φ must be finite as $r \to 0$, so inside the sphere $B_l = 0$

Also, as $r \to \infty$, Φ must not blow up. Thus, outside the sphere $A_l = 0$.

Note, though, that when $l=0$, Φ still doesn't blow up. Thus we allow A_0 to survive.

So, we have:

$$\Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_{out}(r,\theta) = A_{0,out} + \sum_{l=0} B_l r^{-(l+1)} P_l(\cos\theta)$$

Let's find Q_{tot} on the sphere so that I can write down another expression for Φ_{out}. $Q_{tot} = \int \sigma dV = \int_\alpha^\pi \int_0^{2\pi} \frac{Q}{4\pi R^2} \cdot R^2 \sin\theta \, d\theta \, d\phi = \frac{Q}{2}\int_\alpha^\pi \sin\theta \, d\theta = \frac{Q}{2}(1-\cos(\alpha))$

Now, as $r \to \infty$, this sphere "looks" like a pt. charge.

Then $\Phi_{out}(r \to \infty) = \frac{Q_{tot}}{4\pi\varepsilon_0 r} = \frac{Q}{8\pi\varepsilon_0 r}(1-\cos(\alpha))$

We also know what happens to Φ_{out} as $r \to \infty$ by looking at the general expression above:

$\Phi_{out}(r\to\infty) = A_{0,out} \frac{1}{r}B_0 + \cancel{\frac{1}{r^2}B_1\cos\theta + \frac{1}{r^3}B_2\cos\theta + \cdots}$ (we notice that $l \geq 0$ must be dropped b/c $P_l(\cos(\theta))$ for $l \geq 0$ is dependent on θ. This can not be! It's a point charge). $\to$ The $O(\frac{1}{r^2})$ part of each term also ensures this.

Now we set ~~the two~~ the two expressions for Φ_{out} equal to each other.

$A_{0,out} \frac{1}{r} B_0 = \frac{Q}{8\pi\varepsilon_0 r}(1-\cos(\alpha))$

Thus, $A_{0,out} = 0$ and $B_0 = \frac{Q}{8\pi\varepsilon_0} \cdot (1+\cos(\alpha))$

(It makes sense that $A_{0,out} = 0$ b/c there should not be a baseline potential as we approach ∞!)

Now we have: $\Phi_{in}(r,\theta) = A_0 + \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta)$

$$\Phi_{out}(r,\theta) = \frac{Q}{8\pi\varepsilon_0} \cdot (1+\cos\alpha) + \sum_{l=1}^{\infty} (B_l r^{-(l+1)} P_l(\cos\theta))$$

Where I have pulled out A_0.

Now we must find A_l and B_l. To do that we will use two boundary conditions.

$\Phi_{in}(r=R) = \Phi_{out}(r=R) \Rightarrow A_0 + \sum_{\ell=1}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \frac{Q}{8\pi\varepsilon_0}(1+\cos\alpha) + \sum_{\ell=1}^{\infty}(B_\ell R^{-\ell-1} P_\ell(\cos\theta))$

Equating term by term: $A_0 + A_\ell R^\ell P_\ell(\cos\theta) = \frac{Q}{8\pi\varepsilon_0}\cdot(1+\cos\alpha) + B_\ell R^{-\ell-1} P_\ell(\cos\theta)$

Thus $A_{0,in} = \frac{Q}{8\pi\varepsilon_0}\cdot(1+\cos\alpha)$, and $A_\ell R^\ell = B_\ell R^{-\ell-1} \Rightarrow B_\ell = A_\ell R^{2\ell+1}$

Thus, the potentials become:

$\Phi_{in} = \frac{Q}{8\pi\varepsilon_0}(1+\cos\alpha) + \sum_{\ell=1}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

$\Phi_{out} = \frac{Q}{8\pi\varepsilon_0}(1+\cos\alpha) + \sum_{\ell=1}^{\infty} A_\ell R^{2\ell+1} r^{-\ell-1} P_\ell(\cos\theta)$

Now the second Boundary Condition: $\frac{\sigma}{\varepsilon_0} = \left(\frac{\partial}{\partial r}\Phi_{in} - \frac{\partial}{\partial r}\Phi_{out}\right)\Big|_{r=R}$

$\sigma/\varepsilon_0 = \sum_{\ell=1}^{\infty} A_\ell \ell R^{\ell-1} P_\ell(\cos\theta) + \sum_{\ell=1}^{\infty} A_\ell(\ell+1) R^{2\ell+1} R^{-\ell-2} P_\ell(\cos\theta)$

This must be true for all ℓ, so drop summations.

$A_\ell \ell R^{\ell-1} P_\ell(\cos\theta) + A_\ell(\ell+1) R^{\ell-1} P_\ell(\cos\theta) = \sigma/\varepsilon_0$

$A_\ell R^{\ell-1} P_\ell(\cos\theta)[2\ell+1] = \sigma/\varepsilon_0$

Now use Fourier's Trick, (multiply both sides by $P_{\ell'}(\cos\theta)\sin\theta$, and integrate w/ respect to θ.

$\int A_\ell R^{\ell-1}(2\ell+1) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \frac{\sigma}{\varepsilon_0}\int P_{\ell'}(\cos\theta)\sin\theta\, d\theta$

These Legendre Polynomials are orthogonal, so integral is zero, except when $\ell=\ell'$.

Then, $\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2\ell+1}$

$-2A_\ell R^{\ell-1} = Q/(4\pi R^2\varepsilon_0)\int_\alpha^\pi P_\ell(\cos\theta)\sin\theta\, d\theta$ This integral gives $\frac{1}{(2\ell+1)}[P_{\ell-1}(\cos\alpha) - P_{\ell+1}(\cos\alpha)$

So our last unknown is:

$$A_\ell = \frac{Q}{8\pi\varepsilon_0 R^{\ell+1}}\frac{1}{(2\ell+1)}[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)]$$

$$\boxed{\begin{aligned}\Phi_{in}(r,\theta) &= \frac{Q}{8\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\left(P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right)\frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)\\ \Phi_{out}(r,\theta) &= \frac{Q}{8\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\left(P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right)\frac{R^\ell}{r^{\ell+1}} P_\ell(\cos\theta)\end{aligned}}$$

→ What constraint needs to be imposed on $P_{-1}(\cos\alpha)$?

(b) Find $E(r=0)$ $\quad \vec{E}(r=0) = -\nabla(\Phi_{in})|_{r=0} = \left(-\frac{\partial}{\partial r}\Phi_{in}\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}\Phi_{in}\hat{\theta}\right)$

Note that when $\ell=0$, $\frac{\partial}{\partial r}\Phi_{in}=0$, and $P_0(\cos\theta)=1$, so $\frac{\partial}{\partial\theta}\Phi_{in}=0$

Thus, we ignore the $\ell=0$ terms:

$$\vec{E} = \frac{-Q}{8\pi\varepsilon_0}\sum_{\ell=1}^{\infty}\frac{\ell}{2\ell+1}\left(P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right)\frac{r^{\ell-1}}{R^{\ell+1}} P_\ell(\cos\theta)\hat{r}$$
$$- \frac{Q}{8\pi\varepsilon_0}\sum_{\ell=1}^{\infty}\left(\frac{1}{2\ell+1}\right)\left(P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right)\frac{r^{\ell-1}}{R^{\ell+1}}\left(\frac{\partial}{\partial\theta}P_\ell(\cos\theta)\right)\hat{\theta}$$

When $\ell=1$, $\frac{\partial}{\partial\theta}P_1(\cos\theta) = -\sin\theta$. AND for $\ell>1$, $r^\ell=0$ @ $r=0$, so we only worry about the $\ell=1$ term.

$$\vec{E}(r=0) = \frac{-Q}{8\pi\varepsilon_0}\frac{1}{3}\left(P_2(\cos\alpha) - P_0\cancel{(\cos\alpha)}^{\,1}\right)\frac{1}{R^2}P_1\cancel{(\cos\theta)}^{\,\cos\theta}\hat{r}$$
$$+\frac{Q}{8\pi\varepsilon_0}\frac{1}{3}\left(P_2(\cos\alpha) - P_0(\cancel{\cos\alpha})\right)\frac{1}{R^2}\sin\theta\,\hat{\theta}$$
$$\vec{E}(r=0) = \frac{-Q}{24\pi\varepsilon_0}\left(\frac{3\cos^2\alpha - 1}{2} - 1\right)\frac{1}{R^2}(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

Note $\frac{3\cos^2\alpha-1}{2} - 1 = \frac{3\cos^2\alpha - 1 - 2}{2} = \frac{3}{2}(\cos^2\alpha - 1) = \frac{3}{2}\sin^2\alpha$

$$\vec{E}(r=0) = \frac{-Q\sin^2\alpha}{16\pi\varepsilon_0 R^2}(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

and $\hat{r}\cos\theta - \hat{\theta}\sin\theta = \cancel{\cos\theta\sin\theta}\cos\phi\,\hat{x} + \cancel{\sin\theta\cos\theta}\sin\phi\,\hat{y} + \cos^2\theta\,\hat{z}$
$\qquad - \cancel{\cos\theta\sin\theta}\cos\phi\,\hat{x} - \cancel{\sin\theta\cos\theta}\sin\phi\,\hat{y} + \sin^2\theta\,\hat{z} = \hat{z}$

$$\boxed{\vec{E}(r=0) = \frac{+Q\sin^2(\alpha)}{16\pi\varepsilon_0 R^2}\hat{z}}$$

(watch signs ... think about what direction the field should be in, provided that the "spherical cap" is at the top of the sphere.)

(c) $\vec{E}(r=0)$ as $\alpha \to 0$, $\sin\alpha \to \alpha$, so $\vec{E}(r=0) = \frac{+Q\alpha^2}{16\pi\varepsilon_0 R^2}\hat{z}$

Thus as $\alpha \to 0$, $\vec{E}(r=0) \to 0$. Obviously, when the sphere closes up, the field inside goes to zero (you're completely shielded).

as $\alpha \to \pi$ then call $\beta = \pi - \alpha$, and $\beta \to 0$. Thus similarly $\vec{E}(r=0) \to 0$. This also makes sense since the charged sphere is almost completely gone @ this point.

Φ_{in}. As $\alpha \to 0$, $P_\ell(\cos\alpha) \to P_\ell(1)$ and for all ℓ, $P_\ell(1) = 1$

So $P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) \to 0$. Thus, $\Phi_{in} \to 0$ as well as $\alpha \to 0$. This makes sense since again, we are w/in a closed conducting shell.

Now as $\alpha \to \pi$, $P_\ell(\cos(\pi))$ goes to 1 if ℓ is even and -1 iff odd. Thus, for ℓ even $(P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos\alpha)) = 0$ as does ℓ odd. Phew. So we get the same behavior. $\Phi_{in} \to 0$.

Again, this makes sense. We are approaching no charge.

(what happens if you go beyond leading order in α?)