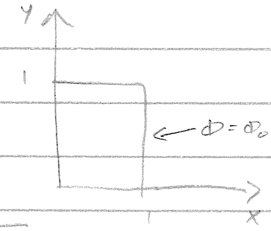
Jackson 2-15 [5] (a) Begin by noting that G will be some linear combination of sin functions.



$G = \sum_{n=1}^{\infty} a_n \sin(n\pi x')$

By def. $\nabla^2 G = -4\pi\,\delta(x-x')\,\delta(y-y')$ plug this in...

$$\sum_n \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) a_n \sin(n\pi x') = -\sum_n \left(-4\pi\,\delta(y-y')\right) \sqrt{\tfrac{2}{1}}\sin(n\pi x)\sqrt{\tfrac{2}{1}}\sin(n\pi x')$$

take derivatives:

$$\sum_n a_n\left((-n^2\pi^2)\sin(n\pi x') + \frac{\partial^2}{\partial y^2}\cancel{\sin(n\pi x')}\right) = \sum_n \left(\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right)\sin(n\pi x')\, a_n\right.$$

This we set equal to $\sum_n 8\pi\,\delta(y-y')\sin(n\pi x)\sin(n\pi x')$

thus $a_n = 2g_n(y',y)\sin(n\pi x)$ (A)

and for each term

$$\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) a_n = -8\pi\,\delta(y-y')\sin(n\pi x)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) 2g_n(y-y')\cancel{\sin(n\pi x)} = -8\pi\,\delta(y-y')\cancel{\sin(n\pi x)}$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) g_n(y-y') = -4\pi\,\delta(y-y')}\ \checkmark$$

Now plug (A) into our initial equation for G:

$$\boxed{G = 2\sum_n g_n(y',y)\sin(n\pi x)\sin(n\pi x')}\ \checkmark$$

(b) We are told that g_n is a linear combo of hyperbolic sinh and cosh.
in general:

$g_{nL}(y,y') = a\sinh(n\pi y') + b\cosh(n\pi y')$ $y' < y$

$g_{nR}(y,y') = c\sinh(n\pi y') + d\cosh(n\pi y')$ $y' > y$

on the boundary, when $y'=0$, then g_{nL} must be zero, so $b=0 \Rightarrow a\sinh(n\pi y')$

also when $y'=1$, then $g_{nR}(y,1) = 0 \Rightarrow c\sinh(n\pi) + d\cosh(n\pi)$

Thus $\frac{1}{2}\left(c(e^{n\pi} - e^{-n\pi}) + d(e^{n\pi} + e^{-n\pi})\right) = 0$

$= \left(\frac{c}{2} + \frac{d}{2}\right)e^{n\pi} + \left(\frac{c}{2} - \frac{d}{2}\right)e^{-n\pi} = 0$

One possibility is for $\left(\frac{c}{2} + \frac{d}{2}\right) = e^{-n\pi}$, $\left(\frac{c}{2} - \frac{d}{2}\right) = e^{n\pi}$

so $c = -d - e^{-n\pi}$ and $c = d + e^{n\pi} \Rightarrow -e^{-n\pi} - e^{n\pi} = 2d$

thus $d = -\cosh(n\pi)$, and similarly $c = \sinh(n\pi)$

Plugging back in for $g_{nR}(y, y')$

$$g_{nR}(y, y') = -\cosh(n\pi)\sinh(n\pi y') + \sinh(n\pi)\cosh(n\pi y')$$
$$= \cosh(n\pi)\sinh(-n\pi y') + \sinh(n\pi)\cosh(-n\pi y')$$
$$= \sinh(n\pi - n\pi y') = \sinh(n\pi(1-y'))$$

We actually suppressed a constant. I'll add it back now.

$$g_{nL} = \lambda \sinh(n\pi y')$$
$$g_{nR} = \gamma \sinh(n\pi(1-y'))$$

When $y=y'$, these equal: $\lambda \sinh(n\pi y) = \gamma \sinh(n\pi(1-y'))$

this can be forced to hold when:

$\lambda = K\sinh(n\pi(1-y'))$ and $\gamma = K\sinh(n\pi y)$

$$g_n = \begin{cases} K_n \sinh(n\pi(1-y))\sinh(n\pi y') & y' < y \\ K_n \sinh(n\pi y)\sinh(n\pi(1-y)) & y' > y \end{cases}$$

Putting these together $g_n(y, y') = K_n \sinh(n\pi(1-y_>))\sinh(n\pi y_<)$

Then: $G(x, y, x', y') = 2\sum_n K \sin(n\pi x)\sin(n\pi x')\sinh(n\pi(1-y_>))\sin(n\pi y_<)$

The jump condition then says

$$\left.\frac{\partial g}{\partial y}\right|_{y_>} - \left.\frac{\partial g}{\partial g}\right|_{y_<} = -4\pi = n\pi K_n[-\cosh(n\pi(1-y))\sinh(n\pi y) - \sinh(n\pi(1-y))\cosh(n\pi y)]$$
$$-n\pi K\sinh(n\pi y)$$

$$K = \frac{-4\pi}{n\pi\sinh(n\pi)} = \frac{4}{n\sinh(n\pi)}$$

Finally

$$\boxed{G(x, y, x', y') = 8\sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi x')\sinh(n\pi(1-y_>))\sinh(n\pi y_<)}{n\sinh(n\pi)}}$$