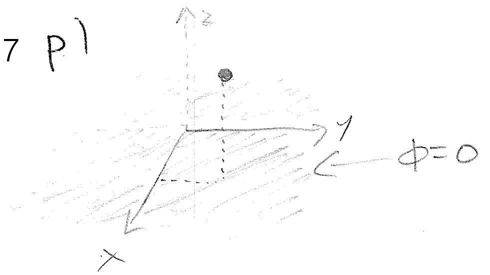
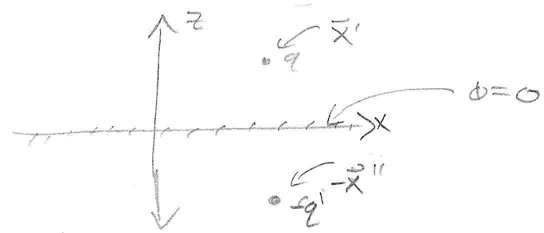
Jackson 2.7 p1



Take a pt. charge at (x, y, z)



(a) same as for a 2D-representation w/ $z(0) = 0$. We can try find ϕ for a pt. charge q in $z > 0$. Simple Method of images problem!

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{x}'|} - \frac{q}{|\vec{x}-\vec{x}''|}\right]$$ where $\vec{x}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$, and $\vec{x}'' = x'\hat{x} + y'\hat{y} - z'\hat{z}$.

$$\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$$

we have Dirichlet Boundary condition, so

$$\Phi_D(\vec{x}) = k\int G_D(\vec{x}-\vec{x}')\,\rho(\vec{x})\,dV' - \frac{1}{4\pi}\oint \Phi_D(\vec{x}')\frac{\partial G_D}{\partial n'}\,ds'$$

$$q\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right] = \int G_D(\vec{x}-\vec{x}')\,\rho(\vec{x})\,dV' - \varepsilon_0 \oint \Phi_D(\vec{x}')\frac{\partial G_D}{\partial n'}\,ds'$$

$\to 0$ vanishes on Surface

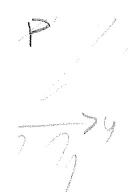
$$\cancel{q}\left[\sim\right] = \cancel{q}\,G_D(\vec{x}-\vec{x}') \Rightarrow \boxed{G(\vec{x}-\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}}$$

(b) ~~Convert Greens function to cylind.~~

{Next time use ρ}

$x^2 - 2xx' + x'^2 + y^2 + y'^2$

$$G(\vec{x}-\vec{x}') = \frac{1}{\sqrt{x^2+y^2+z^2+x'^2+y'^2+z'^2-2xx'-2yy'-2zz'}} - [\ldots] = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\phi-\phi')+(z-z')^2}} - \frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\phi-\phi')+(z+z')^2}}$$



$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{All space}} G(\vec{x}-\vec{x}')\,\rho(\vec{x})\,dV' - \frac{1}{4\pi}\oint_{z=0}\phi(x)\frac{\partial G}{\partial n}\,ds' = \frac{-1}{4\pi}\int_{|r|\le a} V\,\frac{\partial G}{\partial z}\,r'\,dr'\,d\phi$$

$$\Rightarrow \text{find } \frac{\partial G}{\partial z} = \frac{-(z-z')}{\left[r^2+r'^2-2rr'\cos(\phi-\phi')+(z-z')^2\right]^{3/2}} - \frac{-(z+z')}{\left[r^2+r'^2-2rr'\cos(\phi-\phi')+(z+z')^2\right]^{3/2}}$$

$$\text{simp} = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^a V\,\frac{(z-z')\,r'}{\left[\sim\right]^{3/2}}\,dr'\,d\phi' - \frac{1}{4\pi}\int V \text{ etc}\Big|_{z'=0}$$

$$) = \frac{V}{2\pi}\int_0^{2\pi}\!\!\int_0^a \frac{z\,r'}{\left[r^2+r'^2-2rr'\cos(\phi-\phi')+z^2\right]^{3/2}}\,dr'\,d\phi'$$

p2

Prime: source

ⓒ set $\rho=0$: $\phi(\vec{x}) = \frac{Vz}{2\pi}\int_0^a\int_0^{2\pi} \frac{r'}{[r'^2+z^2]^{3/2}}d\phi' dr' = Vz\int_0^a \frac{r'}{(r'^2+z^2)^{3/2}}dr' = \frac{-Vz}{\sqrt{r'^2+z^2}}\Big|_0^a$

$= -Vz\left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{z}\right) = \boxed{V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)}$ ✓

ⓓ $\rho^2+z^2 \gg a^2$ Let $w^2=\rho^2+z^2$

$\phi(\vec{x}) = \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho'}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}}d\rho' d\phi' = \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho'}{[w^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]^{3/2}}d\rho' d\phi$

Binomial Expansion

$\frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a\left(\frac{\rho'}{w^{3/2}}\right)\left(\frac{1}{\left(1+\frac{\rho'^2}{w}-\frac{2\rho\rho'\cos(\phi-\phi')}{w}\right)^{3/2}}\right)d\rho' d\phi'$

use Binomial expansion: $(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2$

$= \frac{Vz}{2\pi w^{3/2}}\int_0^{2\pi}\int_0^a \rho' d\rho' d\phi'\left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{w}\right)^{-3/2} = \frac{Vz}{2\pi w^2}\int_0^{2\pi}\int_0^a \rho' d\rho' d\phi'\left(1-\frac{3(\rho'^2-2\rho\rho'\cos(\phi-\phi'))}{2w}+\frac{15(\rho'^2-2\rho\rho'\cos(\phi-\phi'))^2}{2w^2}\right)$

$= \frac{Vz}{2\pi w^{3/2}}\left[\int_0^{2\pi}\int_0^a \rho' d\rho' d\phi' - \frac{3}{2}\int_0^{2\pi}\int_0^a \frac{\rho'^3}{w}d\rho' d\phi' + \frac{6}{2}\int_0^{2\pi}\int_0^a \frac{\rho'^2\rho\cos(\phi-\phi')}{w}d\rho' d\phi' \xrightarrow{} 0 \quad \frac{15}{8}\int_0^{2\pi}\int_0^a \frac{\rho'^5}{w^2}d\rho' d\phi' + \frac{15}{4}\int_0^{2\pi}\int_0^a \frac{\rho^2\rho'^3\cos^2(\phi-\phi')}{w^2}d\rho' d\phi' - \frac{15}{4}\int_0^{2\pi}\int_0^a \frac{\rho\rho'^4\cos(\phi-\phi')}{w^2}d\rho' d\phi' \xrightarrow{} 0\right]$

$= \frac{Vz}{2\pi w^{3/2}}\left[\pi a^2 - \frac{3}{4}\frac{a^4\pi}{w} + 0 + \frac{5}{8}\frac{a^6\pi}{w^2} + \frac{15a^4\pi\rho^2}{8w^2} - 0\right]$

where we note both terms w/ $\int_0^{2\pi}\cos(\phi)d\phi$ have gone to zero.

$= \frac{Va^2}{2}\frac{z}{w^{3/2}}\left[1-\frac{3}{4}\frac{a^2}{w}+\frac{5}{8}\frac{a^4}{w^2}+\frac{5}{8}\frac{(3a^2\rho^2)}{w^2}\right]$

$\phi(\vec{x}) = \boxed{\frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3}{4}\frac{a}{(\rho^2+z^2)}+\frac{5}{8}\left(\frac{3a^2\rho^2+a^4}{(\rho^2+z^2)^2}\right)\right]}$ ✓

where we have truncated after the first 3 terms.

<u>Check</u>

when $\rho=0$ and $\rho^2+z^2 \gg a^2$, then $z^2 \gg a^2$

$\phi(\vec{x}) = \frac{Va^2}{2}\frac{z}{z^3}\left[1-\underbrace{\frac{3}{4}\frac{a}{z^2}}_{0}+\frac{5}{8}\underbrace{\left(\frac{a^4}{z^4}\right)}_{0}\right] = \underbrace{\frac{Va^2}{2z^2}}_{0} = 0$. From ⓒ, $V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \rightarrow V\left(1-\frac{z}{\sqrt{z^2}}\right) \Rightarrow 0$ ✓ ⑩

Both approximations agree!