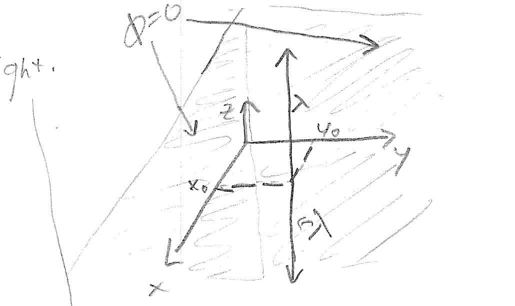
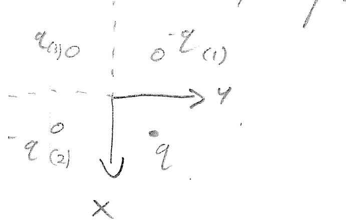
Jackson 2.3 p1

We set this up as a method of images problem, with 3 images charges as shown @ right.



top view by images



(a) $$\phi(x,y)=\frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right)+\ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right)-\ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)-\ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)\right]$$

(★) $$=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{[(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2]}{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}\right] ✓$$

Check @ $x=0$. $$\phi(0,y)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{[\cancel{x_0^2+(y+y_0)^2}][\cancel{x_0^2+(y-y_0)^2}]}{[\cancel{x_0^2+(y-y_0)^2}][\cancel{x_0^2+(y+y_0)^2}]}\right]=0 ✓$$

similarly @ $y=0$, the same things cancel, and $\phi(x,0)=0$ ✓

Now find $E(x,y)=-\vec{\nabla}\phi(x,y)=\frac{\partial\phi}{\partial x}\hat{x}+\frac{\partial\phi}{\partial y}\hat{y}$

$$=\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{-2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}-\frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}+\frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2}+\frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2}\right]\hat{x}$$

$$+\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}-\frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2}+\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}+\frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2}\right]\hat{y}$$

The component tangential to $x=0$ is $(\vec{E}\cdot\hat{y})$. At $x=0$, this is zero

$$\frac{\lambda}{4\pi\varepsilon_0}\left(\cancel{\frac{-2(y-y_0)}{x_0^2+(y-y_0)^2}}-\cancel{\frac{2(y+y_0)}{x_0^2+(y+y_0)^2}}+\cancel{\frac{2(y+y_0)}{x_0^2+(y+y_0)^2}}+\cancel{\frac{2(y-y_0)}{(x_0^2)+(y-y_0)^2}}\right)=0 ✓$$

similarly at $y=0$, the $\vec{E}\cdot\hat{x}=0$. ✓

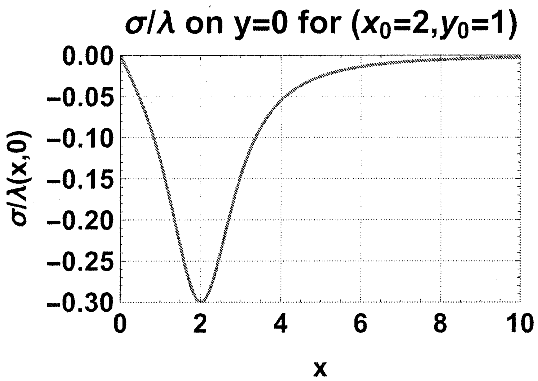
(b) $$\sigma=-\varepsilon_0\frac{\partial\phi}{\partial y}=\frac{-2\lambda y_0}{4\pi}\left[\frac{+1}{(x-x_0)^2+y_0^2}-\frac{1}{(x+x_0)^2+y_0^2}+\frac{1}{(x-x_0)^2+y_0^2}+\frac{-1}{(x+x_0)^2+y_0^2}\right]$$

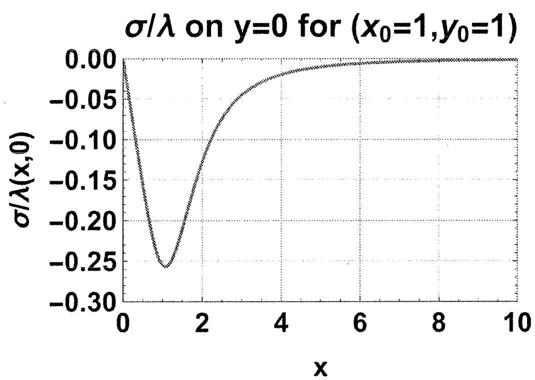
$$\sigma(x)=\frac{-\lambda y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2}-\frac{1}{(x+x_0)^2+y_0^2}\right] ✓$$

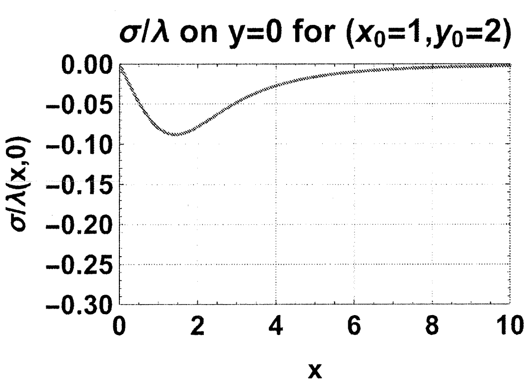
See attached Mathematica plot on the next page for the plots ✓

In[85]:= $$\texttt{sigmaOverLambda[x0\_, y0\_] = } -\frac{\texttt{y0}}{\pi}\left(\frac{1}{(\texttt{x}-\texttt{x0})^2+\texttt{y0}^2}-\frac{1}{(\texttt{x}+\texttt{x0})^2+\texttt{y0}^2}\right);$$

In[95]:=
```
Plot[sigmaOverLambda[2, 1], {x, 0, 10}, PlotRange → {{0, 10}, {-.3, 0}},
  GridLines → Automatic, Frame → True, FrameLabel → {"x", "σ/λ(x,0)"},
  BaseStyle → {FontSize -> 14, Bold}, PlotLabel → "σ/λ on y=0 for (x₀=2,y₀=1)"]
Plot[sigmaOverLambda[1, 1], {x, 0, 10}, PlotRange → {{0, 10}, {-.3, 0}},
  GridLines → Automatic, Frame → True, FrameLabel → {"x", "σ/λ(x,0)"},
  BaseStyle → {FontSize -> 14, Bold}, PlotLabel → "σ/λ on y=0 for (x₀=1,y₀=1)"]
Plot[sigmaOverLambda[1, 2], {x, 0, 10}, PlotRange → {{0, 10}, {-.3, 0}},
  GridLines → Automatic, Frame → True, FrameLabel → {"x", "σ/λ(x,0)"},
  BaseStyle → {FontSize -> 14, Bold}, PlotLabel → "σ/λ on y=0 for (x₀=1,y₀=2)"]
```

Out[95]=


Out[96]=


Out[97]=


P2

(c) $Q = \int \sigma \, da = \int_0^\infty \frac{-\lambda y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right] dx$

$$= \frac{-\lambda y_0}{\pi}\left.\left(\frac{\tan^{-1}\left(\frac{x-x_0}{y_0}\right)}{y_0} - \frac{\tan^{-1}\left(\frac{x+x_0}{y_0}\right)}{y_0}\right)\right|_0^\infty = \frac{-\lambda}{\pi}\left(\tan^{-1}\left(\frac{x-x_0}{y_0}\right) - \tan^{-1}\left(\frac{x+x_0}{y_0}\right)\right)$$

$\tan^{-1}(x) \to \frac{\pi}{2}$ as $x \to \infty$

$$= \frac{-\lambda}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2} - \left(\tan^{-1}\left(\frac{-x_0}{y_0}\right) - \tan^{-1}\left(\frac{x_0}{y_0}\right)\right)\right)$$

$$= \frac{-\lambda}{\pi}\left[-\tan^{-1}\left(\frac{-x_0}{y_0}\right) + \tan^{-1}\left(\frac{x_0}{y_0}\right)\right]$$ and $\tan^{-1}(-x) = -\tan^{-1}(x)$ $\boxed{Q_x = \frac{-2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)}$ as expected

Now, we do the same thing for Q_y. We expect to get $Q_y = -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{y_0}{x_0}\right)$ b/c of interchange symmetry.

$$Q_y = \int \sigma \, da = \int_0^\infty -\varepsilon_0 \frac{\partial \Phi}{\partial x} dy = \int_0^\infty -\left[\frac{\lambda}{4\pi}\left(\frac{-2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2} + \frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2}\right)\right] dy$$

@ $x=0$

$$= \frac{-\lambda}{4\pi}\int_0^\infty \frac{2x_0}{x_0^2+(y-y_0)^2} - \frac{2x_0}{x_0^2+(y+y_0)^2} - \frac{2x_0}{x_0^2+(y+y_0)^2} + \frac{2x_0}{(x_0)^2+(y-y_0)^2}\, dy$$

$$= \frac{-\lambda 2x_0}{4\pi}\int_0^\infty \frac{2}{x_0^2+(y-y_0)^2} - \frac{2}{x_0^2+(y+y_0)^2}\, dy = \frac{-\lambda x_0}{\pi}\int \frac{1}{(y-y_0)^2+x_0^2} - \frac{1}{(y+y_0)^2+x_0}\, dy$$

$$= \frac{-\lambda}{\pi}\left.\left(\tan^{-1}\left(\frac{y-y_0}{x_0}\right) - \tan^{-1}\left(\frac{y+y_0}{x_0}\right)\right)\right|_0^\infty = \frac{-\lambda}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2} - \left(\tan^{-1}\left(\frac{-y_0}{x_0}\right) - \tan^{-1}\left(\frac{y_0}{x_0}\right)\right)\right)$$

$$= \frac{-\lambda}{\pi}\, 2\tan^{-1}\left(\frac{y_0}{x_0}\right) = \boxed{\frac{-2\lambda}{\pi}\tan^{-1}\left(\frac{y_0}{x_0}\right)}$$

Finally, together $Q_x + Q_y = \frac{-2\lambda}{\pi}\left(\tan^{-1}\left(\frac{x_0}{y_0}\right) + \tan^{-1}\left(\frac{y_0}{x_0}\right)\right)$ since $\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x)$

$$= \frac{-2\lambda}{\pi}\left(\frac{\pi}{2} - \cancel{\tan^{-1}\left(\frac{y_0}{x_0}\right)} + \cancel{\tan^{-1}\left(\frac{y_0}{x_0}\right)}\right) = -\lambda$$ as expected!

P3 Expand out equation ☆ from the first page of this problem..

(d) $$\phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x^2+x_0^2+y^2+y_0^2-2xx_0+2yy_0)(x^2+x_0^2+y^2+y_0^2+2xx_0-2yy_0)}{(x^2+x_0^2+y^2+y_0^2-2xx_0-2yy_0)(x^2+x_0^2+y^2+y_0^2+2xx_0+2yy_0)}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\rho^2+\rho_0^2-2xx_0+2yy_0)(\rho^2+\rho_0^2+2xx_0-2yy_0)}{(\rho^2+\rho_0^2-2xx_0-2yy_0)(\rho^2+\rho_0^2+2xx_0+2yy_0)}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\rho^4+2\rho_0^2\rho^2-\cancel{2xx_0\rho^2}+\cancel{2yy_0\rho^2}+\rho_0^4-\cancel{2xx_0\rho_0^2}+\cancel{2yy_0\rho_0^2}-4x^2x_0^2+4yy_0xx_0+4xx_0yy_0-4y^2y_0^2}{\rho^4+2\rho_0^2\rho^2-\cancel{2xx_0\rho^2}-\cancel{2yy_0\rho^2}+\rho_0^4-\cancel{2xx_0\rho_0^2}-\cancel{2yy_0\rho_0^2}-4x^2x_0^2-4yy_0xx_0-4xx_0yy_0-4y^2y_0^2}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\rho^4+2\rho_0^2\rho^2+\rho_0^4-4x^2x_0^2-4y^2y_0^2+8xx_0yy_0}{\rho^4+2\rho_0^2\rho^2+\rho_0^4-4x^2x_0^2-4y^2y_0^2-8xx_0yy_0}\right] = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(1+2\frac{\rho_0^2}{\rho^2}+\frac{\rho_0^4}{\rho^4}-\frac{4x^2x_0^2}{\rho^4}-\frac{4y^2y_0^2}{\rho^4}+\frac{8xxyy_0}{\rho^4}\right)\right.$$

$$\left.-\ln\left(1+\frac{2\rho_0^2}{\rho^2}+\frac{\rho_0^4}{\rho^4}-\frac{4x^2x_0^2}{\rho^4}-\frac{4y^2y_0^2}{\rho^4}-\frac{8xxyy_0}{\rho^4}\right)\right]$$

assuming x_0, y_0 are on order ρ_0, then all but all but the 1st and last terms in the arg of each log are of order 1. Lets focus on them because higher order terms must be very small when $\rho \gg \rho_0$

$$\approx \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(1+\frac{8xx_0yy_0}{\rho^4}\right)-\ln\left(1-\frac{8xx_0yy_0}{\rho^4}\right)\right]$$

and $\ln(x) = (x-1)$ to first order

$$\approx \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{8xx_0yy_0}{\rho^4}-\left(\frac{-8xx_0y_0}{\rho^4}\right)\right] = \boxed{\frac{4\lambda(xy)(x_0y_0)}{\pi\varepsilon_0\rho^4}}$$ ✓

Interpret.

Since it goes to zero like $\frac{1}{\rho^4}$, we can see that this is basically the quadrupole term. It makes sense that this dominates @ large distances as we have the equivalent of four seperated charges in the tell-tale quadrupole allignment: ⊕⊖ / ⊖⊕. ✓ (nice!)

10