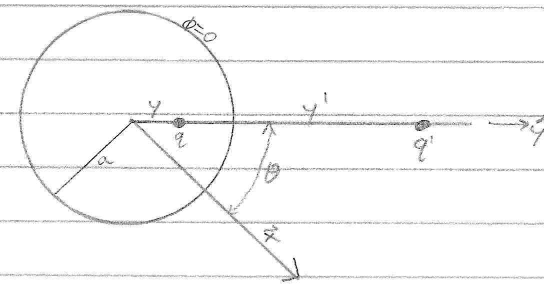
Jackson 2-2

[2] Here we have a charge q Inside a hollow sphere which conducts, and is set @ $\Phi = 0$. We place an image charge q' at $\vec{y} = y'$



ⓐ $$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right)$$

we want $\Phi(|x|=a)=0$, so based on this, find $\Phi(\vec{x})$

Then $$\frac{q}{|\vec{a}-\vec{y}|} = \frac{-q'}{|\vec{a}-\vec{y}'|} \Rightarrow \left(\frac{q}{-q'}\right) = \frac{|\vec{a}-\vec{y}|}{|\vec{a}-\vec{y}'|}$$

$$\left(\frac{q}{q'}\right)^2 = \frac{\vec{a}^2+\vec{y}^2-2\vec{a}\cdot\vec{y}}{\vec{a}^2+(\vec{y}')^2-2\vec{a}\cdot\vec{y}'} = \frac{a^2+y^2-2ay\cos\theta}{a^2+(y')^2-2ay\cos\theta}$$

$$\left(\frac{q}{q'}\right)^2\left(a^2+(y')^2-2ay'\cos\theta\right) - \left(a^2+y^2-2ay\cos\theta\right) = 0$$

$$\left[\left(\frac{q}{q'}\right)^2(-2ay') + 2ay\right]\cos\theta + \left(\frac{q}{q'}\right)^2(a^2+(y')^2) - (a^2+y^2) = 0$$

But we should be able to vary θ w/o changing the answer, so individually the quantity in brackets should independently equal zero.

$$\left(\frac{q}{q'}\right)^2(-2ay') + 2ay = 0 \Rightarrow y' = y\left(\frac{q'}{q}\right)^2$$

$$\Rightarrow q' = \pm q\sqrt{\frac{y'}{y}}$$

of course we take the negative version!

Meanwhile $$\left(\frac{y}{y'}\right)(a^2+y'^2) - (a^2+y^2) = 0$$

$$y(a^2+y'^2) - y'(a^2+y^2) = 0$$

$$\Rightarrow y'^2 - y'\left(\frac{a^2+y^2}{y}\right) + a^2 = 0$$

$$= \left(y'-\frac{a^2}{y}\right)(y'-y) = 0 \Rightarrow y' = \frac{a^2}{y} \text{ or } y' = y$$

Clearly the image charge can't be at y! So $y' = \frac{a^2}{y}$

$$\boxed{\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{x}-\vec{y}|} - \frac{a}{y}\frac{1}{\left|\vec{x}-\left(\frac{a^2}{y^2}\vec{y}\right)\right|}\right) \quad @\ |\vec{x}|<a}$$ ✓

ⓑ $$\sigma = -\varepsilon_0\frac{\partial\Phi}{\partial n}\Big|_{|\vec{x}|=a} = -\varepsilon_0\left(\frac{-\partial\Phi}{\partial\vec{x}}\right)\Big|_{|\vec{x}|=a} = \left(\frac{q}{4\pi}\right)\left(\frac{\partial}{\partial x}\left[\frac{1}{|\vec{x}-\vec{y}|}\right] - \frac{\partial}{\partial x}\left[\frac{a/y}{\left|\vec{x}-\left(\frac{a^2}{y^2}\vec{y}\right)\right|}\right]\right)$$

$$= \frac{q}{4\pi}\left(\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2-2xy\cos\theta}}\right] - \frac{a}{y}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+a^4/y^2-\frac{2xa^2}{y}\cos\theta}}\right]\right)$$

$$= \frac{q}{4\pi}\left(\frac{-1}{2}\frac{2x-2y\cos\theta}{(x^2+y^2-2xy\cos\theta)^{3/2}} + \frac{a}{2y}\left[\frac{(2x-2a^2/y\cos\theta}{(x^2+\frac{a^4}{y^2}-\frac{2xa^2}{y}\cos\theta)^{3/2}}\right.\right.$$

$$= \frac{-q}{8\pi}\left[\frac{2a-2y\cos\theta}{(a^2+y^2-2ay\cos\theta)^{3/2}} - \frac{a}{y}\frac{2a-2a^3/y\cos\theta}{\left(a^2+\frac{a^4}{y^2}-\frac{2a^3}{y}\cos\theta\right)^{3/2}}\right]$$

$$= \frac{-q}{4\pi}\left[\frac{a-y\cos\theta}{y^3\left(\frac{a^2}{y^2}+1-2\frac{a}{y}\cos\theta\right)^{3/2}} - \frac{a-\frac{a^2}{y}\cos\theta}{a^2y\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\theta\right)^{3/2}}\right]$$

$$= \left[\frac{\frac{a}{y^3}-\frac{\cos\theta}{y^2}-\frac{1}{ay}+\frac{\cos\theta}{y^2}}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\theta\right)^{3/2}}\right]\frac{-q}{4\pi} = \frac{-q}{4\pi y^3 a}\left[\frac{a^2-y^2}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\theta\right)^{3/2}}\right]$$

$$= \boxed{\frac{-q}{4\pi y^3 a}\frac{(a^2-y^2)}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\theta\right)^{3/2}}} \checkmark$$

(c) Simply the force from the image acting on it.

$$F_{q' on q} = \frac{1}{4\pi\varepsilon_0}\frac{qq'(\vec{y}-\vec{y}')}{|\vec{y}-\vec{y}'|^3} = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\frac{\vec{y}\left(1-\frac{a^2}{y^2}\right)}{\left|\vec{y}\left(1-\frac{a^2}{y^2}\right)\right|^3} = \frac{-q^2}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\frac{\vec{y}}{\left(1-\frac{a^2}{y^2}\right)^2|\vec{y}|^3}$$

$$\boxed{\vec{F}_{q' on q} = \frac{q^2 a}{4\pi\varepsilon_0 y^3}\left(\frac{1}{\left(1-\frac{a^2}{y^2}\right)^2}\right)\hat{y}} \checkmark$$

always outward except @ center.

(d) Say we add a potential V to the sphere.

Then $\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{x}-\vec{y}|} - \frac{a}{y}\left(\frac{1}{\left|\vec{x}-\left(\frac{a^2}{y^2}\right)\vec{y}\right|}\right)\right) + V$

For σ, we take the derivative, so σ is unchanged.

Potential doesn't come into calc. of Force, so force is unchanged.

(-1) What about the second case?

(9)