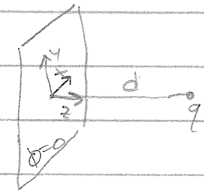
[1] Jackson 2-1

We have the situation drawn @ right → which we can collapse into a really simple Method of Images Problem.

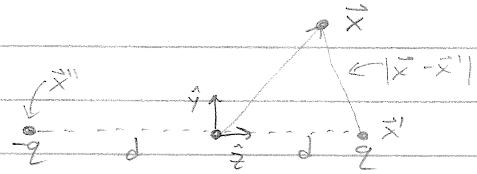


Thus $\vec{x} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{x}' = d\hat{z}$

$\vec{x} - \vec{x}' = x\hat{x} + y\hat{y} + (z-d)\hat{z}$

$|\vec{x} - \vec{x}'| = \sqrt{x^2+y^2+(z-d)^2}$

Similaly, $|\vec{x} - \vec{x}''| = \sqrt{x^2+y^2+(z+d)^2}$



Then, find ϕ

$$\phi(\vec{x}) = \frac{kq}{|\vec{x}-\vec{x}'|} + \frac{k(-q)}{|\vec{x}-\vec{x}''|} = kq\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

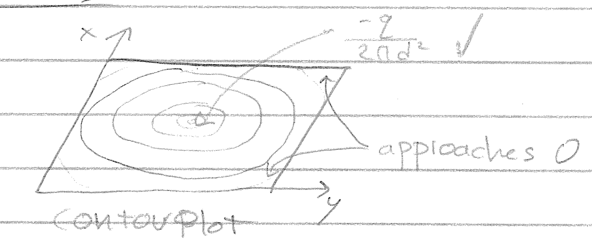
ⓐ $$\sigma = -\varepsilon_0 \frac{\partial\phi}{\partial n} = -\varepsilon_0 \frac{\partial}{\partial z}\left[kq\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)\right]$$

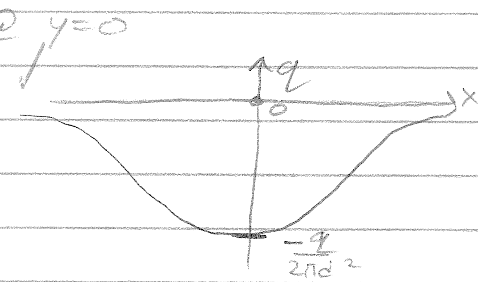
$$= \frac{-q}{4\pi}\left[\frac{-(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{-(z+d)}{(x^2+y^2+(z-d)^2)^{3/2}}\right] = \frac{q}{4\pi}\left[\frac{(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

on the plane $z=0$

$$\sigma = \frac{q}{4\pi}\left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right] = \frac{-q}{2\pi}\left[\frac{d}{(x^2+y^2+d^2)^{3/2}}\right] = \boxed{\frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$$

Plots





ⓑ We can regard the forces of charge on plane as the same as the force of the image charge on the real charge.

$$\vec{F}_{q\,on\,-q} = \frac{kq(-q)\hat{z}}{|\vec{x}-\vec{x}'|^2} = \frac{-q^2k}{|-d\hat{z}-d\hat{z}|^2}\hat{z} = \frac{-kq^2}{(2d)^2} = \boxed{\frac{-q^2}{16\pi\varepsilon_0 d^2}\hat{z}}$$

ⓒ $E = \frac{dF}{da}$

$E = \frac{\sigma^2}{2\varepsilon_0}$ $\quad \frac{dF}{da} = \frac{\sigma^2}{2\varepsilon_0}\hat{z} \Rightarrow dF = \frac{\sigma^2}{2\varepsilon_0}da$

$$\Rightarrow F = \int \frac{\sigma^2}{2\varepsilon_0}dx\,dy\,\hat{z} = \frac{q^2d^2}{4\pi^2\,2\varepsilon_0}\int \frac{1}{(x^2+y^2+d^2)^3}dx\,dy\,\hat{z}$$

$$= \frac{q^2d^2}{8\pi^2\varepsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{1}{(r^2+d^2)^3}r\,dr\,d\theta\,\hat{z} = \frac{q^2d^2}{4\pi\varepsilon_0}\int\frac{r}{(r^2+d^2)^3}dr\,\hat{z}$$

using $u = r^2 + d^2$ so $du = 2r\,dr$

$$= \frac{q^2 d^2}{8\pi\varepsilon_0} \int \frac{du}{u^3}\hat{z} = \frac{-q^2 d^2}{16\pi\varepsilon_0 (d^2 + r^2)^2}\Big|_0^\infty = \boxed{\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{z}}$$

(d) $$W = \int_d^\infty \vec{F} \cdot d\vec{x} = \int_d^\infty \frac{1}{16\pi\varepsilon_0} \frac{q^2}{x^2}\,dx = \frac{-1}{16\pi\varepsilon_0} \frac{q^2}{x}\Big|_d^\infty = \boxed{\frac{q^2}{16\pi\varepsilon_0 d}}$$

(e) $$U_E = (-q)\, k\frac{q}{x} = \frac{-kq^2}{4\pi\varepsilon_0 2d} = \frac{-q^2}{8\pi\varepsilon_0 d}$$

This is double the work to move the charge to ∞ b/c we must move both the charge AND the image together if we treat the images as the real-life situation.

(f) $$W = \frac{(1.60 \cdot 10^{-19}\,C)^2}{16\pi\left(8.85 \cdot 10^{-12}\,\frac{C^2}{N \cdot m^2}\right) \cdot 10^{-10}\,m} \cdot \frac{1\,eV}{1.6 \cdot 10^{-19}\,J} = \boxed{3.6\,eV}$$

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