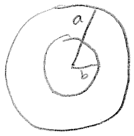
44.2



$a-b \ll a$ (Not drawn to scale).

Reworked now that I understand

This solution is known to be correct

The modes are characterized by $(\nabla_\perp^2 + \gamma^2)\Phi(\rho, \phi) = 0$

Assume $\Phi(\rho,\phi) = \Phi(\rho)\, e^{im\phi}$

Then $(\nabla_\perp^2 + \gamma^2)\Phi(\rho,\phi) = 0 = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \gamma^2\right) e^{im\phi}\,\Phi(\rho) = 0$

$\Rightarrow 0 = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + \gamma^2\right)\Phi(\rho,\phi) = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + \gamma^2\right)\Phi(\rho) = 0$

↖ Dividing out $e^{im\phi}$

Now let $\Phi(\rho) \to \frac{1}{\sqrt{\rho}}\Phi(\rho)$ (ok b/c LHS $=0$).

Note that $\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\frac{1}{\sqrt{\rho}}\Phi(\rho) = \frac{1}{\sqrt{\rho}}\frac{\Phi(\rho)}{4\rho^2} + \frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\Phi(\rho)$ (use Mathematica).

$\Rightarrow 0 = \left[\frac{\partial^2}{\partial\rho^2} - \frac{m^2}{\rho^2} + \frac{1}{4\rho^2} + \gamma^2\right]\sqrt{\rho}\,\Phi(\rho) = 0$ where I allowed $\Phi(\rho) \to \sqrt{\rho}\,\Phi(\rho)$ to get back to where we started.

Next, note that $a-b \ll a$, so $\rho \simeq b$ for all ρ.

$\therefore \left[\frac{\partial^2}{\partial\rho^2} \underbrace{- \frac{m^2-\frac{1}{4}}{b^2} + \gamma^2}_{\equiv\, \omega^2}\right]\Phi(\rho) = 0 \quad \Rightarrow \quad \left[\frac{\partial^2}{\partial\rho^2} + \omega^2\right]\Phi(\rho) = 0$

Let's solve the ODE.

The characteristic equation is $\lambda^2 + \omega^2 = 0 \Rightarrow$ the roots are $\lambda = \pm i\omega$

Thus we have $\Phi(\rho) = A\cos(\omega\rho) + B\sin(\omega\rho)$

Change of variable: $\rho \to \rho - b$. This shifts our origin to the inner edge of the wave guide.

$\Phi(\rho) = A\cos(\omega(\rho-b)) + B\sin(\omega(\rho-b))$

E-mode (TM mode) B.C.'s: $\Phi(a) = \Phi(b) = 0$ (by 44.62).

$\Rightarrow A = 0 \Rightarrow \Phi = B\sin(\omega(\rho-b))$

Also, to make $\Phi(a) = 0 = B\sin(\omega(a-b))$, we require $\omega = \frac{n\pi}{a-b}$ $(n \in \mathbb{Z}^+)$

$\Rightarrow \omega^2 = \frac{n\pi^2}{(a-b)^2} = \gamma^2 - \frac{m^2-\frac{1}{4}}{b^2} \Rightarrow \gamma^2 = \frac{n\pi^2}{(a-b)^2} + \frac{m^2-\frac{1}{4}}{b^2}$

Since $a-b \ll b$, $\gamma^2 \simeq \frac{n\pi^2}{(a-b)^2} \Rightarrow \boxed{\gamma_{cutoff} = \frac{\pi}{a-b}}$ b/c $n_{min} = 1$ for TM mode. (otherwise $\Phi = 0$)

M-mode TE-mode B.C.'s $\Phi'(a) = \Phi'(b) = 0$ (by 44.62).

$\Rightarrow \Phi = A\cos(\omega(\rho-b))$ Also, to make $\Phi'(a) = 0$, we need $\omega^2 = \frac{n^2\pi^2}{(a-b)^2} = \gamma^2 - \frac{m^2-\frac{1}{4}}{b^2}$

$\Rightarrow \gamma^2 = \frac{n\pi^2}{(a-b)^2} + \frac{m^2-\frac{1}{4}}{b^2}$ as before, but for TE waves, $n=0$ is allowed. (Φ = constant, so that's ok).

Thus $\gamma \gtrsim \frac{\sqrt{1-\frac{1}{4}}}{b} \sim \boxed{\frac{\sqrt{3}}{2a} = \gamma_{cutoff}}$ ($m_{min} = 1$ — otherwise γ would be imaginary).