$\boxed{40.3}$ To derive the given expression for $\frac{dP_m}{d\Omega}$, we begin w/ eq 38.42:

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$$\frac{dP_m}{d\Omega} = \frac{\omega_0 e^2}{2\pi R}\beta^3 m^2 \left(\left[J'_m(m\beta\sin\theta) \right]^2 + \left[\frac{J_m(m\beta\sin\theta)}{\beta\tan\theta} \right]^2 \right)$$

By 38.53, $J_m(z) = \int_0^\pi \frac{1}{\pi} d\phi \cos(z\sin\phi - m\phi)$

$$\Rightarrow J_m(m\beta\sin\theta) = \int_0^\pi \frac{1}{\pi} d\phi \cos(m\beta\sin\theta\sin\phi - m\phi)$$

We assume $\omega_0 t = \phi$ is small, so we can taylor expand $\sin\phi \approx \phi - \frac{\phi^3}{6}$

$$\Rightarrow J_m(m\beta\sin\theta) = \int_0^\pi \frac{1}{\pi} d\phi \cos\left(m\beta\phi\sin\theta - \frac{m\beta\sin\theta\,\phi^3}{6} - m\phi\right)$$

$$= \int_0^\pi \frac{1}{\pi} d\phi \cos\left(-m\phi(1-\beta\sin\theta) - \frac{1}{6}m\beta\sin\theta\,\phi^3\right)$$

- note $1-\beta\sin\theta \approx \frac{1}{2}(1-\beta\sin^2\theta) = \frac{1}{2}\varepsilon$ ... (given in the problem)
- note $\cos(-x) = \cos(x)$, so we flip the sign in the argument

$$= \int_0^\pi \frac{1}{\pi} d\phi \cos\left(\frac{m}{2}\phi\varepsilon - \frac{1}{6}m\beta\sin\theta\,\phi^3\right)$$

- for the ultrarelativistic limit, $\theta \to \frac{\pi}{2}$, and $\beta \to 1$

Solution continues on the next page….

thus $J_m(m\beta\sin\theta) \simeq \int_0^{\pi} \frac{1}{\pi} d\phi \cos\left(\frac{m}{2}\left(\phi\varepsilon + \frac{\phi^3}{3}\right)\right)$

- change of variables: $\psi = \phi\left(\frac{m}{2}\right)^{1/3}$
- in the ultrarelativistic limit The lower limit of integration stays at $\psi = (0)\left(\frac{m}{2}\right)^{1/3} = 0$

But, The upper limit goes to $\psi = (\pi)\left(\frac{m}{2}\right)^{1/3}$, which in the ultrelativistic limit $\to \infty$.

$$J_m(m\beta\sin\theta) \simeq \left(\frac{2}{m}\right)^{1/3} \int_0^{\infty} \frac{1}{\pi} d\psi \cos\left(\left(\frac{m}{2}\right)^{2/3}\varepsilon\psi + \frac{1}{3}\psi^3\right)$$

- make the two following definitions:

$$\left(\frac{m}{2}\right)^{2/3}\varepsilon \equiv z$$

$$t \equiv \psi$$

$$J_m(m\beta\sin\theta) \simeq \frac{1}{\sqrt{3}}\sqrt{\frac{3\varepsilon}{z}} \int_0^{\infty} \frac{1}{\pi} dt \cos\left(zt + \frac{t^3}{3}\right) = \frac{\varepsilon}{\pi\sqrt{3}} K_{1/3}(\xi)$$

by the definition of $K_{1/3}$ in the problem.

- next, $z = \left(\frac{3\xi}{2}\right)^{2/3}$ from the problem statment, so

$$\xi = \frac{2z^{3/2}}{3} = \frac{2}{3}\left(\frac{m}{2}\right)\varepsilon^{3/2} = \frac{m}{3}\varepsilon^{3/2}$$

$$\Rightarrow J_m(m\beta\sin\theta) = \frac{\varepsilon}{\pi\sqrt{3}} K_{1/3}\left(\frac{m}{3}\varepsilon^{3/2}\right)$$

We plug this into The second term in paretheses of the original equation:

$$\left[\frac{J_m(m\beta\sin\theta)}{\beta\tan\theta}\right]^2 = \frac{\varepsilon K_{1/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right)}{\pi^2 3\beta^2\tan^2\theta} \simeq \frac{\varepsilon\cos^2\theta\, K_{1/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right)}{\pi^2 3}$$

where I sent $\beta \to 1$, and $\tan\theta \simeq \sin\theta = \frac{1}{\cos\theta}$ in the small angle aprox.

Other term (very similar, so I'll go faster):

$$J'_m(z) = \frac{d}{dz}\int_0^{\pi} \frac{1}{\pi} d\phi \cos(z\sin\phi - m\phi) = -\int_0^{\pi} \frac{d\phi}{\pi} \sin\phi \sin(z\sin\phi - m\phi)$$

small angle approx $\simeq -\int_0^{\pi} \frac{d\phi}{\pi} \phi \sin(z\sin\phi - m\phi)$

using the same argument as before,

$$J'_m(m\beta\sin\theta) \simeq \int_0^{\pi} \frac{d\phi}{\pi} \phi \sin\left(\frac{m}{2}\left(\varepsilon\phi + \frac{\phi^3}{3}\right)\right)$$

again, let $\psi = \phi\left(\frac{m}{2}\right)^{1/3}$, and the upper limit of integration $\left(\frac{m}{2}\right)^{1/3}\pi \to \infty$

$$\Rightarrow J'_m(m\beta\sin\theta) = \left(\frac{2}{m}\right)^{2/3} \int_0^{\infty} \frac{1}{\pi} d\psi \sin\left[\left(\frac{m}{2}\right)^{2/3}\varepsilon\psi + \frac{\psi^3}{3}\right]$$

now $\psi \to t$ and $\left(\frac{m}{2}\right)^{2/3}\varepsilon \to z$

$$= \frac{1}{\pi}\frac{\varepsilon}{z}\int_0^{\infty} dt\, t \sin\left(tz + \frac{1}{3}t^3\right) = \frac{\varepsilon}{\pi\sqrt{3}}\frac{\sqrt{3}}{z}\int_0^{\infty} dt\, t \sin\left(zt + \frac{t^3}{3}\right)$$

$$= \frac{\varepsilon}{\pi\sqrt{3}} K_{2/3}(\xi) = \frac{\varepsilon}{3\sqrt{\pi}} K_{2/3}\left(\frac{m}{3}\varepsilon^{3/2}\right)$$

$$\Rightarrow \left[J'_m(m\beta\sin\theta)\right]^2 \simeq \frac{\varepsilon^2}{3\pi^2} K_{2/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right)$$

$$\Rightarrow \frac{dP_m}{d\Omega} = \frac{\omega_0 e^2}{2\pi R} m^2 \beta^{\cancel{3}\to 1}\left[\frac{\varepsilon^2}{3\pi^2} K_{2/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right) + \frac{\varepsilon}{\pi^2 3}\cos^2\theta\, K_{1/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right)\right]$$

$$\Rightarrow \boxed{\frac{dP_m}{d\Omega} = \omega_0 \frac{e^2 m^2}{R 6\pi^3}\left[\varepsilon^2 K_{2/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right) + \varepsilon\cos^2\theta\, K_{1/3}^2\left(\frac{m}{3}\varepsilon^{3/2}\right)\right]}$$