31.5 We'll derive the scalar potential first.

31.49) $\Phi(\vec{r},t) = \int d\vec{r}'dt' \frac{\delta(\frac{1}{c}|\vec{r}-\vec{r}'|-(t-t'))}{|\vec{r}-\vec{r}'|} e\delta(\vec{r}'-\vec{r}(t'))$ ✓ (plugging in the given charge density)

Integrate over $\vec{r}'$

$$= e\int dt' \frac{\delta(\frac{1}{c}|\vec{r}-\vec{r}(t')|-(t-t'))}{|\vec{r}-\vec{r}(t')|}$$ ✓

+3/3

Now let $f(t') = \frac{1}{c}|\vec{r}-\vec{r}(t')| - (t-t')$ and $g(t') = \frac{1}{|\vec{r}-\vec{r}(t')|}$

$$= e\int dt' \delta(f(t'))g(t')$$

Next note that $\int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_i \frac{g(x_i)}{f'(x_i)}$ ✓ (where the x_i are roots of f)

The root of $f(t')$ are at $t'(0)=t$, so

$$\Phi = e\frac{g(t')}{f'(t')}$$ ← that's the derivative

Now take the derivative.., $f' = \frac{\partial}{\partial t'}f(t') = \frac{1}{c}\frac{\partial}{\partial t'}\sqrt{(\vec{r}-\vec{r}(t'))^2} + 1$

$$f' = \frac{1}{c}\frac{(\frac{d}{dt}(-\vec{r}(t')))(\vec{r}-\vec{r}(t'))}{\sqrt{(\vec{r}-\vec{r}(t'))^2}} + 1$$

$$f' = 1 - \frac{\vec{v}(t')\cdot(\vec{r}-\vec{r}(t'))}{c|\vec{r}-\vec{r}(t')|}$$ ✓ (b/c $-\frac{d}{dt}\vec{r}(t') = \vec{v}(t')$)

So we plug it in:

✓ $\Phi(\vec{r},t) = e\left(\frac{1}{\cancel{|\vec{r}-\vec{r}(t')|}}\right)\frac{\cancel{|\vec{r}-\vec{r}(t')|}}{(|\vec{r}-\vec{r}(t')| - \frac{\vec{v}(t')}{c}\cdot(\vec{r}-\vec{r}(t')))}$

$$\boxed{\Phi(\vec{r},t) = \frac{e}{|\vec{r}-\vec{r}(t')| - (\vec{r}-\vec{r}(t'))\cdot\frac{\vec{v}(t')}{c}}}$$

Now the vector potential.

31.50) ✓ $\vec{A}(\vec{r},t) = \int d\vec{r}'dt \frac{\delta(\frac{1}{c}|\vec{r}-\vec{r}'|-(t-t'))e\vec{v}(t')\delta(\vec{r}'-\vec{r}(t'))}{c|\vec{r}-\vec{r}'|}$ (plugging in the given current density)

Integrate over $\vec{r}'$

$$= \frac{e}{c}\int dt \frac{\delta(\frac{1}{c}|\vec{r}-\vec{r}(t')-(t-t'))}{|\vec{r}-\vec{r}(t')|}\vec{v}(t')$$

✓

Now let $f(t') = \frac{1}{c}|\vec{r}-\vec{r}(t')|-(t-t')$, and $g(t') = \frac{\vec{v}(t')}{|\vec{r}-\vec{r}(t')}$

We use exactly the same trick as above, and end up with

✓ $\vec{A}(\vec{r},t) = \frac{e}{c}\frac{g(t')}{f'(t')} = \frac{\vec{v}(t')}{c}\left[e\left(\frac{1}{\cancel{|\vec{r}-\vec{r}(t')|}}\right)\frac{\cancel{|\vec{r}-\vec{r}(t')|}}{(|\vec{r}-\vec{r}(t')-\frac{\vec{v}(t')}{c}(\vec{r}-\vec{r}(t'))}\right] = \boxed{\frac{\vec{v}(t')}{c}\Phi(\vec{r},t)}$