2.1 Begin w/ Maxwell's equations w/ magnetic charge:

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$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{J}_e$ $\qquad \vec{\nabla}\cdot\vec{E} = 4\pi\rho_e$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \frac{4\pi}{c}\vec{J}_m$ $\qquad \vec{\nabla}\cdot\vec{B} = 4\pi\rho_m$

Define $\vec{F} \equiv \vec{E} + i\vec{B}$ and $\vec{J} \equiv \vec{J}_e + i\vec{J}_m$, $\rho \equiv \rho_e + i\rho_m$

→ write maxwell's equations in terms of $\vec{F}$, $\vec{J}$, and ρ:

$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot\vec{E} + i\vec{\nabla}\cdot\vec{B} = 4\pi\rho_e + 4\pi i\rho_m = 4\pi(\rho_e + i\rho_m)$

$\therefore \boxed{\vec{\nabla}\cdot\vec{F} = 4\pi\rho}$

$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{E} + i\vec{\nabla}\times\vec{B} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \frac{4\pi}{c}\vec{J}_m + \frac{i}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi i}{c}\vec{J}_e$

we can multiply through by $1 = -i^2$

$= i\left[\frac{i}{c}\frac{\partial}{\partial t}\vec{B} + \frac{4\pi i}{c}\vec{J}_m + \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{J}_e\right]$

$= i\left[\frac{4\pi}{c}(\vec{J}_e + i\vec{J}_m) + \frac{1}{c}\frac{\partial}{\partial t}(\vec{E} + i\vec{B})\right]$

$\therefore \boxed{\vec{\nabla}\times\vec{F} = \frac{4\pi i}{c}\vec{J} + \frac{i}{c}\frac{\partial}{\partial t}\vec{F}}$

→ Let $\vec{F} \to e^{-i\phi}\vec{F}$, where ϕ is an arbitrary constant.

Show that these two new maxwell's equations retain their form under this Transformation.

$\vec{\nabla}\cdot\vec{F} \to \vec{\nabla}\cdot(e^{-i\phi}\vec{F}) = e^{-i\phi}(\vec{\nabla}\cdot\vec{F}) + \vec{F}(\underbrace{\vec{\nabla}e^{-i\phi}}_{0})$ for example by Griffith's Product Rule (5)

$\vec{\nabla}\times\vec{F} \to \vec{\nabla}\times(e^{-i\phi}\vec{F}) = e^{-i\phi}(\vec{\nabla}\times\vec{F}) - \vec{F}(\vec{\nabla}e^{-i\phi})$ by Griffith's Prod. Rule (7)

Thus $\boxed{\vec{\nabla}\cdot\vec{F} \to e^{-i\phi}(4\pi\rho) \quad \text{and} \quad \vec{\nabla}\times\vec{F} \to e^{-i\phi}\left(\frac{4\pi i}{c}\vec{J} + \frac{i}{c}\frac{\partial}{\partial t}\vec{F}\right)}$

as expected.

→ Express this as a transformation of $\vec{E}$, $\vec{B}$, $\vec{J}$, and ρ

when $\vec{F} \to e^{-i\phi}\vec{F}$, then $\vec{E} + i\vec{B} \to e^{-i\phi}(\vec{E} + i\vec{B}) = e^{-i\phi}\vec{E} + ie^{-i\phi}\vec{B}$

$= \vec{E}\cos\phi - i\vec{E}\sin\phi + i\vec{B}\cos\phi + \vec{B}\sin\phi$

Then $\begin{cases} \vec{E} = Re(\vec{F}) \longrightarrow Re(e^{-i\phi}\vec{F}) \\ \vec{B} = Im(\vec{F}) \longrightarrow Im(e^{-i\phi}\vec{F}) \end{cases}$ so $\boxed{\begin{array}{l} \vec{E} \to \vec{E}\cos\phi + \vec{B}\sin\phi \\ \vec{B} \to -\vec{E}\sin\phi + B\cos\phi \end{array}}$

Now for $\vec{J}$ and ρ, we'll take the divergence

$\vec{\nabla}\cdot\vec{E} \to \vec{\nabla}\cdot\vec{E}\cos\phi + \vec{\nabla}\cdot\vec{B}\sin\phi = \rho_e\cos\phi + \rho_m\sin\phi$

$\vec{\nabla}\cdot\vec{B} \to \vec{\nabla}\cdot(-\vec{E})\sin\phi + \vec{\nabla}\cdot\vec{B}\cos\phi = -\rho_e\sin\phi + \rho_m\cos\phi$

$\Rightarrow \boxed{\begin{array}{l} \rho_e = \rho_e\cos\phi + \rho_m\sin\phi \\ \rho_m = -\rho_e\sin\phi + \rho_m\cos\phi \end{array}}$

Similaly

$$\vec{\nabla}\times\vec{E} \longrightarrow \vec{\nabla}\times\vec{E}\cos\phi + \vec{\nabla}\times\vec{B}\sin\phi$$

$$\vec{\nabla}\times\vec{B} \longrightarrow \vec{\nabla}\times(-\vec{E})\sin\phi + \vec{\nabla}\times\vec{B}\cos\phi$$

$$\Rightarrow \boxed{\begin{aligned} -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \frac{4\pi}{c}\vec{J}_m &\rightarrow \left(-\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \frac{4\pi}{c}\vec{J}_m\right)\cos\phi + \left(\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{J}_e\right)\sin\phi \\ \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{J}_e &\longrightarrow \left(\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \frac{4\pi}{c}\vec{J}_m\right)\sin\phi + \left(\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{J}_e\right)\cos\phi \end{aligned}}$$

Geomtrically, this is like introducing a phase factor wherein each componet oscillates sinusoidally betwen electric $\vdots$ magneic versions.

when $\phi = \frac{\pi}{2}$, $\vec{E} \rightarrow \vec{B}$, $\vec{B} \rightarrow -\vec{E}$, $\rho_e \rightarrow \rho_m$, $\rho_m \rightarrow -\rho_e$

and $\vec{J}_m \rightarrow -\vec{J}_e$, $\vec{J}_e \rightarrow \vec{J}_m$

This is the $\boxed{\text{Duality transformation.}}$