10.10 (a) I will identify the compoents of $F_{\mu\nu}$, then ${}^*F_{\mu\nu}$:

$F_{ij} = \partial_i A_j - \partial_j A_i$

Then let's start with $F_{0i} = \partial_0 A_i - \partial_i A_0 = +\frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \Phi}{\partial x_i} = -E_i$

$\Rightarrow \boxed{F_{0i} = E_i}$

+1.5/2

next $F_{ij} = \partial_i A_j - \partial_j A_i = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = (\nabla \times A)_k \, \varepsilon_{ijk}$

(because $(\vec{\nabla}\times\vec{G})_k = \varepsilon_{klm} \nabla_l F_m$

$\Rightarrow \boxed{F_{ij} = \varepsilon_{ijk} B_k}$ ✓

${}^*F_{0i} = \frac{1}{2}\varepsilon_{0i\alpha\beta} F^{\alpha\beta}$ from class,

$i=1$: ${}^*F_{01} = \frac{1}{2}\varepsilon_{01\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{0123}F^{23} + \varepsilon_{0132}F^{32}) = \frac{1}{2}(F^{23} - F^{32})$

$= \frac{1}{2}(\varepsilon_{231}B^1 - \varepsilon_{321}B^1) = B_1$

$i=2$: ${}^*F_{02} = \frac{1}{2}\varepsilon_{02\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{0213}F^{13} + \varepsilon_{0231}F^{31}) = \frac{1}{2}(-F^{13} + F^{31}) = B_2$

$i=3$: ${}^*F_{03} = \frac{1}{2}\varepsilon_{03\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{0312}F^{12} + \varepsilon_{0321}F^{21}) = \frac{1}{2}(F^{12} - F^{21}) = B_3$

$\Rightarrow \boxed{{}^*F_{0i} = B^i}$ ✓

${}^*F_{ij} = \frac{1}{2}\varepsilon_{ij\alpha\beta}F^{\alpha\beta}$

$i=2, j=3$: ${}^*F_{23} = \frac{1}{2}\varepsilon_{23\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{2301}F^{01} + \varepsilon_{2310}F^{10})$

$= \frac{1}{2}(F^{01} - F^{10}) = \frac{1}{2}\left((\partial^0 A^1 - \partial^1 A^0) - (\partial^1 A^0 - \partial^0 A^1)\right)$

$= \frac{1}{2}\left[\left(-\frac{1}{c}\frac{\partial}{\partial t}A_x - \frac{\partial \Phi}{\partial x}\right) - \left(\frac{\partial \Phi}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}A_x\right)\right]$

$= -\frac{1}{c}\frac{\partial}{\partial t}A_x - \frac{\partial \Phi}{\partial x} = E_x = E_1$

$i=1, j=3$: ${}^*F_{13} = \frac{1}{2}\varepsilon_{13\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{1320}F^{20} + \varepsilon_{1302}F^{02})$

$= \frac{1}{2}(\partial^2 A^0 - \partial^0 A^2 - \partial^0 A^2 + \partial^2 A^0) = \partial^2 A^0 - \partial^0 A^2$

$= \frac{\partial \Phi}{\partial y} + \frac{1}{c}\frac{\partial}{\partial t}A^2 = -E_2$

$i=1, j=2$: ${}^*F_{12} = \frac{1}{2}\varepsilon_{12\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(\varepsilon_{1230}F^{30} + \varepsilon_{1203}F^{03})$

$= \frac{1}{2}(-\partial^3 A^0 + \partial^0 A^3 + \partial^0 A^3 - \partial^3 A^0)$

$= \partial^0 A^3 - \partial^3 A^0 = -\frac{1}{c}\frac{\partial}{\partial t}A^3 - \frac{\partial \Phi}{\partial z} = E_3$

$\Rightarrow \boxed{{}^*F_{ij} = \varepsilon_{ijk}E_k}$ ✓

(b) not for turning in

(c) See attached Mathematica Docmnt

10.10, cont. (d) We seek to reproduce the following Maxwell equations:

(1) $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$ (3) $-\vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0$

(2) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ (4) $\vec{\nabla}\cdot\vec{B} = 0$

$\frac{+2}{2}$

Inhomogeneous

$\rightarrow \vec{\nabla}\cdot\vec{E} = \partial_x E_x + \partial_y E_y + \frac{\partial}{\partial z} E_z = \partial_\nu E^\nu = \partial_\nu F^{0\nu}$

$\rightarrow \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \underbrace{\left(\frac{\partial}{\partial x_i}B^j - \frac{\partial}{\partial x_j}B^i\right)}_{\nu=1,2,3} - \underbrace{\frac{1}{c}\frac{\partial}{\partial t}E_i}_{\nu=0} = \partial_\nu F^{\mu\nu}$ as we saw in part (a)

(because $\partial_i = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$)

So, the first Maxwell law is

$$\boxed{\partial_\nu F^{\mu\nu} = \frac{4\pi}{c}J^\mu}$$ (note $J^0 = c\rho$)

- Sign error would have shown up here since $F_{0i} = -E_i$
$\Rightarrow F^{0i} = E_i$

Homogeneous:

$\rightarrow \vec{\nabla}\cdot\vec{B} = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = \partial_\nu B^\nu = \partial_\nu {}^*F^{0i}$

$\rightarrow -\vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial}{\partial t}\vec{B} = -(\vec{\nabla}\times\vec{E})_\nu - \partial_0 {}^*F^{0i}$

$= -\underbrace{\partial_\nu}_{1,2,3} {}^*F^{\mu\nu} - \partial_0 {}^*F^{0i} = -\partial_\nu {}^*F^{\mu\nu}$

So all together $\boxed{\partial_\nu {}^*F^{\mu\nu} = 0}$

Electrodynamics II Homework 1 — Ben Levy

#3 Schwinger Problem 10.10c

+2
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2

Define $F^{\mu\nu}$ and $F_{\mu\nu}$ as Fup and Fdown respectivly. (The components of these tensors come from Jackson 11.137)

```
Fup = {{0, -Ex, -Ey, -Ez}, {Ex, 0, -Bz, By}, {Ey, Bz, 0, -Bx}, {Ez, -By, Bx, 0}};
Fdown = {{0, Ex, Ey, Ez}, {-Ex, 0, -Bz, By}, {-Ey, Bz, 0, -Bx}, {-Ez, -By, Bx, 0}};
```

Then we contract the two tensors by multiply them component-wize, and then summing up all those products:

```
Total[Total[Fup * Fdown] // FullSimplify]
```

$2\,\mathrm{Bx}^2 + 2\,\mathrm{By}^2 + 2\,\mathrm{Bz}^2 - 2\,\mathrm{Ex}^2 - 2\,\mathrm{Ey}^2 - 2\,\mathrm{Ez}^2$

Thus we have $F^{\mu\nu}F_{\mu\nu} = 2\left(B^2 - E^2\right)$

√ $\boxed{\Rightarrow E^2 - B^2 = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu}}$

Now we must define $^*F_{\mu\nu}$

```
starFdown = {{0, Bx, By, Bz}, {-Bx, 0, Ez, -Ey}, {-By, -Ez, 0, Ex}, {-Bz, Ey, -Ex, 0}};
Total[Total[Fup * starFdown] // FullSimplify]
```

$-4\,\mathrm{Bx}\,\mathrm{Ex} - 4\,\mathrm{By}\,\mathrm{Ey} - 4\,\mathrm{Bz}\,\mathrm{Ez}$

Thus we have $F^{\mu\nu}\ {}^*F_{\mu\nu} = -4\left(\vec{E}\cdot\vec{B}\right)$

$\boxed{\Rightarrow \vec{E}\cdot\vec{B} = +\frac{1}{4}F_{\mu\nu}\,{}^*F_{\mu\nu}}$

Supposed to be $\vec{E}\cdot\vec{B} = \frac{1}{4}F_{\mu\nu}\,{}^*F^{\mu\nu}$ but - sign came from earlier so no points off